Exponential Mixing for 3D Stochastic Primitive Equations of the Large Scale Ocean

Zhao Dong[1,], Jianliang Zhai[2,], Rangrang Zhang[1,]

[1] Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55 Zhongguancun East Road, Haidian District, Beijing, 100190, P. R. China

[2] School of Mathematical Sciences, University of Science and Technology of China, No. 96 Jinzhai Road, Hefei, 230026, P. R. China

(dzhao@amt.ac.cn, zhaijl@ustc.edu.cn, rrzhang@amss.ac.cn)

Abstract

This paper reports weak solutions of 3D stochastic primitive equations of the large scale ocean converge exponentially fast to the equilibrium, if the noise is at the same time sufficiently smooth and non-degenerate. Furthermore, thanks to global well-posedness of the strong solution, weak solutions which are limits of Galerkin approximations share the same invariant measure. In particular, the invariant measure for the strong solution is unique.

1 Introduction

The large-scale motion of the ocean can be well modeled by 3D viscous primitive equations. Beyond their considerable significance in physical applications, the primitive equations have generated much interest from the mathematics community due to their rich nonlinear, nonlocal character and their anisotropic structure.

The mathematical study of the 3D determine primitive equations of the large scale ocean originated in a series of articles by J. L. Lions, R. Temam, and S. Wang in the early 1990s ([18, 19, 20, 21]). They defined the notions of weak and strong solutions and also proved the existence of weak solution. However, the uniqueness of weak solutions in the general case is still an open question, even for the two-dimensional case. In [10], the existence of strong solutions to the primitive equations with small initial data was established. In [13], the authors proved the global existence and uniqueness of strong solution to the primitive equations under the small depth hypothesis for a large set of initial data whose size depends on the thickness of the domain. In [2], C. Cao and E.S. Titi developed a beautiful approach to dealing with the L^6-norm of the fluctuation $\tilde{v}$ of horizontal velocity, and obtained the global well-posedness for the 3D viscous primitive equations.

Despite these great successful developments for the deterministic primitive equations, introducing uncertainty in 'exact' model equations is reasonable and necessary in the study of the primitive equations of the large scale ocean. The introducing of stochastic processes is aimed at accounting for a number of uncertainties and errors. For example, the external forcing of the ocean comes mainly from the atmosphere, and atmospheric forcing field must be regarded as random, see, e.g. [8, 11, 23, 28] and references therein. On the other hand, as mentioned in [6], these 'exact' models are numerically intractable, they can't be fully solved with present super computers(and will not be for any foreseeable future). We refer to e.g. [6] and [11] for more details about the motivations for introducing uncertainty.

Recently, there are several works on the global well-posedness and the long-time behavior of the 3D stochastic primitive equations. In [6], A. Debussche, N. Glatt-Holtz, R. Temam and M. Ziane established the global existence and uniqueness of strong solution for a nonlinear, multiplicative white noise in 2011. For the long-time behavior of the stochastic case, in [24], T. Tachim Medjo obtained the weak solution converges exponentially in the mean square and almost surely exponentially to the stationary solution. In his paper, the viscosity is assumed large enough and the covariance operator of the noise is required to satisfy some exponential decline property. In [11], B. Guo and D. Huang obtained the global well-posedness of strong solution and the existence of global random attractor under the assumptions that the momentum equation is driven by an additive stochastic forcing and the thermodynamical equation is driven by a fixed heat source. In [9], the authors established the continuity of the Markovian semigroup associated with strong solution of the 3D stochastic primitive equations, and proved the existence of invariant measure for the strong solution.

In the present paper, we devote to proving the exponential mixing property for 3D stochastic primitive equations driven by multiplicative noise. The difficulty we encounter is that in both determine and random cases, the uniqueness of weak solutions which start from L^2 space is unknown. Thanks to the idea of [26], we can overcome this difficulty by dealing with weak solutions which are limits of the Galerkin approximations.

Firstly, we need to show the existence of weak solutions of 3D stochastic primitive equations. Since the 3D stochastic primitive equation (2.1)-(2.4) is driven by multiplicative noise, which can not be translated into random parameter partial differential equations, so the method in [11] can not be applied. Here, our proof is based on [7], where compactness method is applied. Further, we consider the exponential mixing property for weak solutions which are limits of Galerkin approximations of 3D stochastic primitive equations. Our proof relies on the modified coupling method from [26], which aims at proving the exponential mixing for the 3D stochastic Navier-Stokes equations. In [26], the author chose the smooth small ball for the entering of two Markov processes is in $\mathbb{H}_2$, however, $\mathbb{H}_2$ is not suitable for our equations, here, we have to choose a smooth small ball in $\mathbb{H}_3$, since our nonlinear term $-\int_{-1}^{z}\nabla_H\cdot v(t,x,y,z')dz'\frac{\partial v}{\partial z}$ is one order higher than $v\frac{\partial v}{\partial z}$. Moreover, during the prove that the time of entering the smooth small ball admits an exponential moment, we have to deal with higher order Sobolev norms than [26], such as $\|\cdot\|^6,\ \|\cdot\|_2^4,\ \|\cdot\|_2^6,\ \|\cdot\|_3^4$. This is highly non-trivial. Last but not least, it is worth mentioning that

with the help of the uniqueness of strong solution of our equations, we can obtain that weak solutions which are limits of Galerkin approximations share the same invariant measure. In particular, the invariant measure for the strong solution is unique (see Corollary 3.3). Such properties are not valid for 3D stochastic Navier-Stokes equations because of the lack of the uniqueness of strong solution, that is, the invariant measure for weak solutions of 3D stochastic Navier-Stokes have to depend on the initial data and Galerkin approximation subsequence (for the detail, see [26]).

This paper is organized as follows. In Sects. 2 and 3, we introduce the 3D stochastic primitive equations and review some basic representational results. Hypothesises on the stochastic forcing are given in Sect.4. The concrete proof of our main results are given in Sects. 5 and 6.

2 Preliminaries

The 3D Stochastic Primitive Equations of the large-scale ocean under a stochastic forcing, in a Cartesian system, are written as

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v\cdot\nabla_H)v + \theta\frac{\partial v}{\partial z} + fk\times v + \nabla_H P - \Delta v &= \sigma_1(v,T)\frac{dW_1}{dt}, && (2.1)\\
\partial_z P + T &= 0, && (2.2)\\
\nabla_H\cdot v + \partial_z\theta &= 0, && (2.3)\\
\frac{\partial T}{\partial t} + (v\cdot\nabla_H)T + \theta\frac{\partial T}{\partial z} - \Delta T &= \sigma_2(v,T)\frac{dW_2}{dt}, && (2.4)
\end{aligned}$$

where the horizontal velocity field $v = (v_1, v_2)$, the three-dimensional velocity field (v_1, v_2, θ), the temperature T and the pressure P are unknown functions. f is the Coriolis parameter. k is vertical unit vector. Set $\nabla_H = (\partial x, \partial y)$ to be the horizontal gradient operator and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ to be the three dimensional Laplacian. W_1 and W_2 are two independent cylindrical Wiener processes on H_1 and H_2, respectively. H_1 and H_2 will be defined below.

The spatial variable (x, y, z) belongs to $\mathcal{M} \triangleq \mathbb{T}^2 \times (-1, 0)$. For simplicity of the presentation, all the physical parameters (height, viscosity, size of periodic box) are set to 1.

Refer to [3], the boundary value conditions for (2.1)-(2.4) are given by

$$v,\ \theta \textit{ and } T \textit{ are periodic in } x \textit{ and } y, \tag{2.5}$$

$$(\partial_z v, \theta)\mid_{z=-1,0} = (0,0),\ T\mid_{z=-1} = 1,\ T\mid_{z=0} = 0. \tag{2.6}$$

$$(v, T)\mid_{t=0} = (v_0, T_0). \tag{2.7}$$

Replacing T and P by $T + z$ and $P - \frac{z^2}{2}$, respectively, then (2.1)-(2.4) with (2.6)-(2.7) is equivalent to the

following system

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v\cdot\nabla_H)v + \theta\frac{\partial v}{\partial z} + fk\times v + \nabla_H P - \Delta v &= \sigma_1(v, T+z)\frac{dW_1}{dt}, && (2.8)\\
\partial_z P + T &= 0, && (2.9)\\
\nabla_H\cdot v + \partial_z\theta &= 0, && (2.10)\\
\frac{\partial T}{\partial t} + (v\cdot\nabla_H)T + \theta(\frac{\partial T}{\partial z}+1) - \Delta T &= \sigma_2(v, T+z)\frac{dW_2}{dt}, && (2.11)
\end{aligned}$$

subject to the boundary and initial conditions

$$v,\ \theta \ and\ T\ are\ periodic\ in\ x\ and\ y, \tag{2.12}$$

$$(\partial_z v, \theta)\mid_{z=-1,0} = (0,0),\ T\mid_{z=-1,z=0} = 0, \tag{2.13}$$

$$(v, T)\mid_{t=0} = (v_0, T_0). \tag{2.14}$$

where we still denote by T_0 the initial temperature in (2.14), though it is now different from that in (2.7).

Inherent symmetries in the equations show that the solution of the primitive equations on $\mathbb{T}^2\times(-1,0)$ with boundaries (2.13)-(2.14) may be recovered by solving the equations with periodic boundary conditions in x,y and z variables on the extended domain $\mathbb{T}^2\times(-1,1)\triangleq\mathbb{T}^3$, and restricting to $z\in(-1,0)$.

To see this, consider any solution of (2.8)-(2.11) with boundaries (2.13)-(2.14), we perform that

$$\begin{aligned}
v(x,y,z) &= v(x,y,-z),\ \text{for (x, y, z)} \in \mathbb{T}^2\times(0,1),\\
T(x,y,z) &= -T(x,y,-z),\ \text{for (x, y, z)} \in \mathbb{T}^2\times(0,1),\\
P(x,y,z) &= P(x,y,-z),\ \text{for (x, y, z)} \in \mathbb{T}^2\times(0,1),\\
\theta(x,y,z) &= -\theta(x,y,-z),\ \text{for (x, y, z)} \in \mathbb{T}^2\times(0,1).
\end{aligned}$$

We also extend σ_1 in the even fashion and σ_2 in the odd fashion across $\mathbb{T}^2\times\{0\}$. Hence, we consider the primitive equations on the extended domain $\mathbb{T}^3=\mathbb{T}^2\times(-1,1)$,

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v\cdot\nabla_H)v + \theta\frac{\partial v}{\partial z} + fk\times v + \nabla_H P - \Delta v &= \sigma_1(v, T)\frac{dW_1}{dt}, && (2.15)\\
\partial_z P + T &= 0, && (2.16)\\
\nabla_H\cdot v + \partial_z\theta &= 0, && (2.17)\\
\frac{\partial T}{\partial t} + (v\cdot\nabla_H)T + \theta(\frac{\partial T}{\partial z}+1) - \Delta T &= \sigma_2(v, T+z)\frac{dW_2}{dt}, && (2.18)
\end{aligned}$$

subject to the boundary and initial conditions

$$v,\ \theta\ , P\ and\ T\ are\ periodic\ in\ x\ ,\ y,\ z, \tag{2.19}$$

$$v\ and\ P\ are\ even\ in\ z,\ \theta\ and\ T\ are\ odd\ in\ z, \tag{2.20}$$

$$(v, T)\mid_{t=0} = (v_0, T_0), \tag{2.21}$$

where T_0 to denote the initial temperature in (2.14), though it is now different from that in (2.7).

Because of the equivalent of the above two kinds of boundary and initial conditions, we consider, throughout this paper, the system (2.15)-(2.21). Note that condition (2.21) is a symmetry condition, which is preserved by system (2.15)-(2.18), that is if a smooth solution to system (2.15)-(2.18) exists and is unique, then it must satisfy the symmetry condition (2.21), as long as it is initially satisfied.

2.1 Reformulation

Note that the vertical velocity θ can be expressed in terms of the horizonal velocity v, through the incompressibility condition (2.17) and the symmetry condition (2.21), as

$$\theta(t,x,y,z) = \Phi(v)(t,x,y,z) = -\int_{-1}^{z} \nabla_H \cdot v(t,x,y,z')dz', \tag{2.22}$$

moreover,

$$\int_{-1}^{1} \nabla_H \cdot v dz = 0.$$

Supposing that p_b is a certain unknown function at Γ_b, and integrating (2.18) from -1 to z, we have

$$P(x,y,z,t) = p_b(x,y,t) - \int_{-1}^{z} T(x,y,z',t)dz'.$$

Then, we make scaling transformation of T. Let $S = \sqrt{C_0}T$, where C_0 is a positive constant will be described in Sect. 4, (2.15)-(2.21) can be rewritten as

$$\frac{\partial v}{\partial t} + (v\cdot\nabla_H)v + \Phi(v)\frac{\partial v}{\partial z} + fk\times v + \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H S\,dz' - \Delta v = \phi(v,S), \tag{2.23}$$

$$\frac{\partial S}{\partial t} + (v\cdot\nabla_H)S + \Phi(v)\frac{\partial S}{\partial z} + \Phi(v) - \Delta S = \varphi(v,S), \tag{2.24}$$

$$\int_{-1}^{1}\nabla_H\cdot v dz = 0, \tag{2.25}$$

where

$$\phi(v,S) = \sigma_1(v, \frac{1}{\sqrt{C_0}}S)\frac{dW_1}{dt}, \quad \varphi(v,S) = \sqrt{C_0}\sigma_2(v, \frac{1}{\sqrt{C_0}}S + z)\frac{dW_2}{dt}.$$

The boundary and initial conditions for (2.23)-(2.25) are given by

$$v \text{ and } S \text{ are periodic in } x,\ y \text{ and } z, \tag{2.26}$$

$$v \text{ and } P \text{ are even in } z,\ \theta \text{ and } T \text{ are odd in } z, \tag{2.27}$$

$$(v,S)\mid_{t=0} = (v_0, S_0). \tag{2.28}$$

In the following, our aim is to prove exponential mixing of (v,S) of (2.23)-(2.28), which implies exponential mixing of the original solution (v,T) of the system (2.15)-(2.21).

2.2 Working Spaces

Let $\mathcal{L}(K_1,K_2)$ (resp. $\mathcal{L}_2(K_1,K_2)$) be the space of bounded (resp. Hilbert-Schmidt) linear operators from the Hilbert space K_1 to K_2. We denote by $|\cdot|_{L^2(\mathbb{T}^2)}$ and $|\cdot|_{H^p(\mathbb{T}^2)}$ the norms of $L^2(\mathbb{T}^2)$ and $H^p(\mathbb{T}^2)$,

respectively. Let $|\cdot|_p$ be the norm of $L^p(\mathbb{T}^3)$ for integer number $p \in (0,\infty)$. In particular, $|\cdot|$ and $(\cdot,\cdot)$ represent the norm and inner product of $L^2(\mathbb{T}^3)$. For the classical Sobolev space $H^p(\mathbb{T}^3)$, $p \in \mathbb{N}_+$,

$$\begin{cases} H^p(\mathbb{T}^3) = \left\{ Y \in L^2(\mathbb{T}^3) \middle| \partial_\alpha Y \in L^2(\mathbb{T}^3) \text{ for } |\alpha| \leq p \right\}, \\ |Y|^2_{H^p(\mathbb{T}^3)} = \sum_{0\leq|\alpha|\leq p} |\partial_\alpha Y|^2. \end{cases}$$

It's known that $(H^p(\mathbb{T}^3), |\cdot|_{H^p(\mathbb{T}^3)})$ is a Hilbert space.

Define working spaces for equations (2.23)-(2.28). Let

$$\mathcal{V}_1 \triangleq \left\{ v \in (C^\infty(\mathbb{T}^3))^2;\ \int_{-1}^{1} \nabla_H \cdot v dz = 0, v \textit{ is periodic in } x,\ y \textit{ and even in } z, \int_{\mathbb{T}^3} v dxdydz = 0 \right\},$$
$$\mathcal{V}_2 \triangleq \left\{ S \in C^\infty(\mathbb{T}^3);\ \ S \textit{ is periodic in } x,\ y \textit{ and odd in } z, \int_{\mathbb{T}^3} S\, dxdydz = 0 \right\},$$

V_1= the closure of $\mathcal{V}_1$ with respect to the norm $|\cdot|_{H^1(\mathbb{T}^3)} \times |\cdot|_{H^1(\mathbb{T}^3)}$,
V_2= the closure of $\mathcal{V}_2$ with respect to the norm $|\cdot|_{H^1(\mathbb{T}^3)}$,
H_1= the closure of $\mathcal{V}_1$ with respect to the norm $|\cdot| \times |\cdot|$,
H_2= the closure of $\mathcal{V}_2$ with respect to the norm $|\cdot|$,

$$V = V_1 \times V_2, \quad H = H_1 \times H_2.$$

The inner products and norms on V, H are given by

$$\begin{aligned} (Y, Y_1)_V &= (v, v_1)_{V_1} + (S, S_1)_{V_2}, \\ (Y, Y_1) &= (v, v_1) + (S, S_1) = (v^{(1)}, (v_1)^{(1)}) + (v^{(2)}, (v_1)^{(2)}) + (S, S_1), \\ (Y, Y)^{\frac{1}{2}}_V &= (v, v)^{\frac{1}{2}}_{V_1} + (S, S)^{\frac{1}{2}}_{V_2}, \quad \|Y\|_V = (Y, Y)^{\frac{1}{2}}_V. \end{aligned}$$

where $Y = (v, S), Y_1 = (v_1, S_1)$. By the Riesz representation theorem, we can identify the dual space H' of H (respectively $H'_i = H_i$, i=1,2). Then we have

$$V \subset H = H' \subset V',$$

where the two inclusions are compact continuous.

2.3 Some Functionals

Define three bilinear forms $a : V \times V \to \mathbb{R}$, $a_1 : V_1 \times V_1 \to \mathbb{R}$, $a_2 : V_2 \times V_2 \to \mathbb{R}$, and their corresponding linear operators $A : V \to V'$, $A_1 : V_1 \to V'_1$, $A_2 : V_2 \to V'_2$ by setting

$$a(Y, Y_1) \triangleq (AY, Y_1) = a_1(v, v_1) + a_2(S, S_1),$$

where

$$a_1(v, v_1) \triangleq (A_1 v, v_1) = \int_{\mathbb{T}^3} \left(\nabla_H v \cdot \nabla_H v_1 + \frac{\partial v}{\partial z} \cdot \frac{\partial v_1}{\partial z} \right) dxdydz,$$

$$a_2(S,S_1) \triangleq (A_2S,S_1) = \int_{\mathbb{T}^3} \left(\nabla_H S \cdot \nabla_H S_1 + \frac{\partial S}{\partial z}\frac{\partial S_1}{\partial z} \right) dxdydz,$$

for any $Y = (v,S)$, $Y_1 = (v_1,S_1) \in V$. The following lemma follows Lemma 2.4 in [19] readily.

Lemma 2.1. **(i)** *The forms a, a_i $(i = 1,2)$ are coercive, continuous, and therefore, the operators $A : V \to V'$ and $A_i : V_i \to V_i'$ $(i = 1,2)$ are isomorphisms. Moreover,*

$$\begin{aligned} a(Y,Y_1) &\leq C_1\|Y\|_V\|Y_1\|_V, \\ a(Y,Y) &\geq C_2\|Y\|_V^2, \end{aligned}$$

where C_1 and C_2 are two positive constants and can be determined in concrete conditions.

(ii) *The isomorphism $A : V \to V'$ (respectively $A_i : V_i \to V_i'$ $(i = 1,2)$) can be extended to a self-adjoint unbounded linear operator on H (respectively on H_i, i=1,2), with compact inverse $A^{-1} : H \to H$ (respectively $A_i^{-1}; H_i \to H_i$ $(i = 1,2)$).*

It's known that A_1 is a self-adjoint operator with discrete spectrum in H_1. Denote by $\{k_n\}_{n=1,2,\cdots}$ an eigenbasis of H_1 associated to the increasing sequence $\{\nu_n\}_{n=1,2,\cdots}$ of eigenvalues of A_1. Similarly, A_2 is a self-adjoint operator with discrete spectrum in H_2. Let $(l_n)_{n=1,2,\cdots}$ be an eigenbasis of H_2 associated to the increasing sequence $\{\lambda_n\}_{n=1,2,\cdots}$ of eigenvalues of A_2. Let $\overline{e}_{n,0} = \begin{pmatrix} k_n \\ 0 \end{pmatrix}$ and $\overline{e}_{0,m} = \begin{pmatrix} 0 \\ l_m \end{pmatrix}$, then we can rearrange $\{\overline{e}_{n,0}, \overline{e}_{0,m}\}$ and denoted it by $\{e_n\}_{n=1,2,\cdots}$, the associated eigenvalues of $(A, D(A))$ is denoted by an increasing sequence $\{\mu_n\}_{n=1,2,\cdots}$.

The fractional power $(A^s, D(A^s))$ of the operator $(A, D(A))$ for $s \in \mathbb{R}$ is

$$\begin{cases} D(A^s) = \{Y = \sum_{n=1}^{\infty} y_n e_n;\ \sum_{n=1}^{\infty} \mu_n^{2s}|y_n|^2 < \infty\}; \\ A^sY = \sum_{n=1}^{\infty} \mu_n^s y_n e_n, \quad \textit{where } Y = \sum_{n=1}^{\infty} y_n e_n. \end{cases}$$

For any $s \in \mathbb{R}$, set

$$\|Y\|_s^A = |A^{\frac{s}{2}}Y|, \quad \mathbb{H}_s^A = D(A^{\frac{s}{2}}).$$

It's obvious that $(\mathbb{H}_s^A, \|\cdot\|_s^A)$ is a Hilbert space and that $(\mathbb{H}_0, \|\cdot\|_0) = (H, |\cdot|)$ and $(\mathbb{H}_1, \|\cdot\|_1) = (V, \|\cdot\|)$. Similarly, define $(\mathbb{H}_s^{A_1}, \|\cdot\|_s^{A_1})$ and $(\mathbb{H}_s^{A_2}, \|\cdot\|_s^{A_2})$, for ease of notations, all of them are denoted by $(\mathbb{H}_s, \|\cdot\|_s)$.

Now, we define three functionals $b : V \times V \times V \to \mathbb{R}$, $b_i : V_1 \times V_i \times V_i \to \mathbb{R}$ $(i = 1,2)$ and the associated operators $B : V \times V \to V'$, $B_i : V_1 \times V_i \to V_i'$ $(i = 1,2)$ by setting

$$\begin{aligned} b(Y,Y_1,Y_2) &\triangleq (B(Y,Y_1),Y_2) = b_1(v,v_1,v_2) + b_2(v,S_1,S_2), \\ b_1(v,v_1,v_2) &\triangleq (B_1(v,v_1),v_2) = \int_{\mathbb{T}^3} \left[(v \cdot \nabla_H)v_1 + \Phi(v)\frac{\partial v_1}{\partial z} \right] \cdot v_2 dxdydz, \\ b_2(v,S_1,S_2) &\triangleq (B_2(v,S_1),S_2) = \int_{\mathbb{T}^3} \left[(v \cdot \nabla_H)S_1 + \Phi(v)\frac{\partial S_1}{\partial z} \right] S_2 dxdydz, \end{aligned}$$

for any $Y = (v,S)$, $Y_i = (v_i,S_i) \in V$. Then we have

Lemma 2.2. *For any $Y \in V$, $Y_1 \in V$,*

$$(B(Y,Y_1),Y_1) = b(Y,Y_1,Y_1) = b_1(v,v_1,v_1) = b_2(v,S_1,S_1) = 0.$$

Moreover, we define another functional $g : V\times V \to \mathbb{R}$ and the associated linear operator $G : V \to V'$ by

$$\begin{aligned} g(Y,Y_1) &\triangleq (G(Y),Y_1) \\ &= \int_{\mathbb{T}^3}\left[f(k\times v)\cdot v_1 + (\nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H S\,dz')\cdot v_1 + \Phi(v)\cdot S_1\right]dxdydz. \end{aligned}$$

By (2.25), we have

$$(v,\nabla_H p_b) = \left(\int_{-1}^{0} vdz, \nabla_H p_b\right)_{L^2(\mathbb{T}^2)} = -\left(p_b, \int_{-1}^{0}\nabla_H\cdot vdz\right)_{L^2(\mathbb{T}^2)} = 0,$$

and since $(v, fk\times v) = 0$, then

Lemma 2.3. (i)

$$g(Y,Y) = (G(Y),Y) = -\frac{1}{\sqrt{C_0}}\int_{\mathbb{T}^3}\Big[\left(\int_{-1}^{z}\nabla_H S\,dz'\right)\cdot v + \Phi(v)\cdot S\Big]dxdydz.$$

(ii) *There exists a constant C, such that*

$$|(G(Y),Y)| \leq C(|S|\|v\| \vee \|S\||v|), \tag{2.29}$$

$$|(G(Y),Y_1)| \leq C|v||v_1| + C(|S|\|v_1\| \vee \|S\||v_1|). \tag{2.30}$$

We combine (2.23) and (2.24), and use the functionals defined in the above to obtain the stochastic evolution equation

$$\begin{cases} dY(t) + AY(t)dt + B(Y(t),Y(t))dt + G(Y(t))dt = \Psi(Y(t))dW(t), \\ Y(0) = y_0. \end{cases} \tag{2.31}$$

where

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \Psi(Y) = \begin{pmatrix} \phi(v,S) & 0 \\ 0 & \varphi(v,S) \end{pmatrix}.$$

2.4 Some inequalities

We recall some interpolation inequalities used later (see [1]).
For $h \in H^1(\mathbb{T}^2)$,

$$\begin{aligned} |h|_{L^4(\mathbb{T}^2)} &\leq c|h|^{\frac{1}{2}}_{L^2(\mathbb{T}^2)}|h|^{\frac{1}{2}}_{H^1(\mathbb{T}^2)}, \\ |h|_{L^5(\mathbb{T}^2)} &\leq c|h|^{\frac{3}{5}}_{L^3(\mathbb{T}^2)}|h|^{\frac{2}{5}}_{H^1(\mathbb{T}^2)}, \\ |h|_{L^6(\mathbb{T}^2)} &\leq c|h|^{\frac{2}{3}}_{L^4(\mathbb{T}^2)}|h|^{\frac{1}{3}}_{H^1(\mathbb{T}^2)}. \end{aligned}$$

For $h \in H^1(\mathbb{T}^3)$,

$$\begin{aligned} |h|_3 &\leq c|h|^{\frac{1}{2}}|h|^{\frac{1}{2}}_{H^1(\mathbb{T}^3)}, \\ |h|_6 &\leq c|h|_{H^1(\mathbb{T}^3)}, \\ |h|_\infty &\leq c|h|^{\frac{1}{2}}_{H^1(\mathbb{T}^3)}|h|^{\frac{1}{2}}_{H^2(\mathbb{T}^3)}. \end{aligned}$$

At last, we recall the integral version of Minkowshy inequality for the L^p spaces, $p \geq 1$. Let $\mathbb{T}^3_1 \subset \mathbb{R}^{m_1}$ and $\mathbb{T}^3_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are two positive integers. Suppose that $f(\xi, \eta)$ is measurable over $\mathbb{T}^3_1 \times \mathbb{T}^3_2$. Then

$$\left[\int_{\mathbb{T}^3_1}\left(\int_{\mathbb{T}^3_2}|f(\xi,\eta)|d\eta\right)^p d\xi\right]^{1/p} \leq \int_{\mathbb{T}^3_2}\left(\int_{\mathbb{T}^3_1}|f(\xi,\eta)|^p d\xi\right)^{1/p} d\eta.$$

Lemma 2.4. *(see [2]) If $v_1 \in H^1(\mathbb{T}^3), v_2 \in H^3(\mathbb{T}^3), v_3 \in H^3(\mathbb{T}^3)$, then*

(i) $|\int_{\mathbb{T}^3} v_3 \cdot [(v_1 \cdot \nabla_H)v_2]dxdydz| \leq c|\nabla v_2||v_3|_3|v_1|_6 \leq c|\nabla v_2||v_3|^{\frac{1}{2}}|\nabla v_3|^{\frac{1}{2}}|\nabla v_1|$,

(ii) $|\int_{\mathbb{T}^3} \Phi(v_1)v_{2z} \cdot v_3 dxdydz| \leq c|\nabla v_1||v_3|^{\frac{1}{2}}|\nabla v_3|^{\frac{1}{2}}|\partial_z v_2|^{\frac{1}{2}}|\nabla \partial_z v_2|^{\frac{1}{2}}$.

2.5 Definition of Weak Solution

Definition 2.1. *We say that there exists a weak solution of equation (2.31) if for any $T > 0$, there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, a cylindrical Wiener process W on H and a progressively measurable process $Y : [0,T] \times \Omega \to H$ such that*

$$Y \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap C\left(0,T;D(A^{-\frac{\alpha}{2}})\right), \quad \mathbb{P}-a.s,$$

where $\alpha > 3$ is any fixed positive number, and for all $\psi \in D(A^{\frac{\alpha}{2}})$, the identity

$$\begin{aligned} (Y(t),\psi) &+ \int_0^t \Big(AY(s),\psi\Big)ds + \int_0^t \Big(B(Y(s),Y(s)),\psi\Big)ds + \int_0^t \Big(G(Y(s)),\psi\Big)ds \\ &= (y_0,\psi) + \int_0^t \Big(\Psi(Y(s))dW(s),\psi\Big), \quad \forall t \in [0,T], \ \mathbb{P}-a.s. \end{aligned}$$

holds.

If $\mathcal{D}(Y(0)) = \lambda \in \mathcal{P}(H)$, we denote $\mathbb{P}_\lambda = \mathcal{D}(Y)$. Here $\mathcal{D}(Z)$ stands for the law of random variable Z.

3 Hypothesis and Main Results

3.1 Hypothesis

Suppose $C_0 \geq \frac{8}{\lambda_1}$ in the preliminaries. The covariance operator Ψ satisfies the following **Hypothesis H0** and **Hypothesis H1**.

Hypothesis H0 $\Psi : H \to \mathcal{L}_2(H, H)$ is a continuous and bounded Lipschitz mapping, i.e.

$$|\Psi(y)|^2_{\mathcal{L}_2(H,H)} \leq \lambda_0 |y|^2 + \rho \quad y \in H,$$

for some constants $\lambda_0 \geq 0,\ \rho \geq 0$.

Hypothesis H1 W_1 and W_2 are two independent cylindrical Wiener processes on H_1 and H_2, respectively.

(1) There exists $\varepsilon_0 > 0$ and a family $\{\Psi_n\}_{n=1,2,\cdots}$ of continuous mappings $H \to \mathbb{R}$ with continuous Frechet derivatives such that

$$\begin{cases} \Psi(y)dW = \sum_{n=1}^{\infty} \Psi_n(y) e_n dW^n \quad where \quad W = \sum_{n=1}^{\infty} W^n e_n, \\ \kappa_0 = \sum_{n=1}^{\infty} \sup_{y \in H} |\Psi_n(y)|^2 \mu_n^{2+\varepsilon_0} < \infty. \end{cases}$$

(2) There exists κ_1 such that for any $y, \eta \in \mathbb{H}_3$,

$$\sum_{n=1}^{\infty} |\Psi_n'(y) \cdot \eta|^2 \mu_n^3 < \kappa_1 \|\eta\|_3^2.$$

(3) For any $y \in H$ and $n \in \mathbb{N}$,

$$\Psi_n(y) > 0, \quad \kappa_2 = \sup_{y \in H} |\Psi^{-1}(y)|^2_{\mathcal{L}(\mathbb{H}_4;H)} < \infty,$$

where

$$\Psi^{-1}(y) \cdot h = \sum_{n=1}^{\infty} \Psi_n^{-1}(y) h_n e_n \quad \text{for} \quad h = \sum_{n=1}^{\infty} h_n e_n.$$

Setting $\kappa = \kappa_0 + \kappa_1 + \kappa_2 + 1$.

Remark 1. *It is easy to know **Hypothesis H1** implies **Hypothesis H0** and $\Psi = A^{-\frac{\beta}{2}}$ fulfills **Hypothesis H1**, provided $\beta \in (\frac{7}{2}, 4]$.*

3.2 Main Results

Our results are as follows.

Theorem 3.1. *[**Existence of Weak Solution**] Under **Hypothesis H0**, there exists a weak solution $Y(t)$ of (2.31) in the sense of Definition 2.1.*

Theorem 3.2. *[**Exponential Convergence**] Assume that **Hypothesis H1** holds. There exist C and $\gamma > 0$ only depending on Ψ, $\mathbb{T}^3$, ε_0 such that, for any weak solution $\mathbb{P}_\lambda$ with initial law $\lambda \in \mathcal{P}(H)$ which is a limit*

of Galerkin approximations of (4.33), there exists a unique stationary measure weak solution $\mu \in \mathcal{P}(H)$*, such that*

$$\|\mathcal{D}_{\mathbb{P}_\lambda}(Y(t)) - \mu\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_H |y|^2 \lambda(dy)\right), \tag{3.32}$$

provided

$$\int_H |y|^2 \lambda(dy) < \infty,$$

where $\mathcal{P}(H)$ *is the set of all probability measures on* H *and* $\|\cdot\|_{var}$ *is the total variation norm associated to the space* $\mathbb{H}_s$ *for* $s < -3$*.*

Remark 2. *In Theorem 3.2, the stationary measure* μ *seems to depend is still unknown on* λ *and the Galerkin approximation subsequence* $\{N_k\}$*, however, with the aid of the uniqueness of strong solution, we can prove that* μ *is independent of* λ *and* $\{N_k\}$ *.*

Corollary 3.3. *Suppose* ***Hypothesis H0*** *and* ***Hypothesis H1*** *hold, then weak solutions which are limits of Galerkin approximations share the same invariant measure, that is,* μ *in Theorem 3.2 is independent of* λ *and* $\{N_k\}$*. In particular, the invariant probability measure for the strong solution is unique.*

4 Existence of Weak Solutions

Our proof is based on [7], compactness method is applied.

4.1 Compact Embedding of Certain Functional Spaces

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis (with expectation $\mathbb{E}$) and W be a cylindrical Wiener process with values in H defined on the stochastic basis. Given $p > 1, \alpha \in (0,1)$, let $W^{\alpha,p}(0,T;H)$ be the Sobolev space of all $u \in L^p(0,T;H)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+\alpha p}} dtds < \infty,$$

endowed with the norm

$$|u|^p_{W^{\alpha,p}(0,T;H)} = \int_0^T |u(t)|^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+\alpha p}} dtds.$$

For any progressively measurable process $f \in L^2(\Omega \times [0,T]; \mathcal{L}_2(H,H))$ denote by $I(f)$ the Itô integral defined as

$$I(f)(t) = \int_0^t f(s)dW(s) \quad t \in [0,T].$$

Clearly, $I(f)$ is a progressively measurable process in $L^2(\Omega \times [0,T]; H)$.

Lemma 4.1. *Let $p \geq 2, \alpha < \frac{1}{2}$ be given. Then for any progressively measurable process $f \in L^p(\Omega \times [0,T]; \mathcal{L}_2(H,H))$,*

$$I(f) \in L^p(\Omega; W^{\alpha,p}(0,T;H))$$

and there exists a constant $C(p,\alpha) > 0$ independent of f such that

$$\mathbb{E}|I(f)|^p_{W^{\alpha,p}(0,T;H)} \leq C(p,\alpha)\mathbb{E}\int_0^T |f(t)|^p_{\mathcal{L}_2(H,H)}dt.$$

Lemma 4.2. *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be given. Let X be the space*

$$X = L^p(0,T;B_0) \cap W^{\alpha,p}(0,T;B_1),$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

Lemma 4.3. *If $B_1 \subset \tilde{B}$ are two Banach spaces with compact embedding, and the real number $\alpha \in (0,1)$, $p > 1$ satisfy $\alpha p > 1$, then the space $W^{\alpha,p}(0,T;B_1)$ is compactly embedded into $C(0,T;\tilde{B})$.*

4.2 Proof of Theorem 3.1.

We divide into three steps to prove this theorem. **Step 1.** Let P_n be the operator from $D(A^{-\frac{3}{2}})$ to $D(A^{\frac{3}{2}})$ defined as

$$P_n x = \sum_{i=1}^{n}\langle x, e_i\rangle e_i \quad x \in D(A^{-\frac{3}{2}}).$$

Here, we denote by $\langle\cdot,\cdot\rangle$ the dual pairing between $D(A^{\frac{3}{2}})$ and $D(A^{-\frac{3}{2}})$. Then

$$\langle P_n x, y\rangle = \langle x, P_n y\rangle,$$

for all $x, y \in D(A^{-\frac{3}{2}})$. Its restriction to H is the orthogonal projection onto $P_nH \triangleq \mathrm{Span}\{e_1,\cdots,e_n\}$. Let $B_n(Y,Y)$ be the Lipschitz operator in P_nH defined as

$$B_n(Y,Y) = \chi_n(Y)B(Y,Y) \quad Y \in P_nH,$$

where $\chi_n : H \to \mathbb{R}$ is defined as $\chi_n(Y) = \Theta_n(|Y|)$, with $\Theta_n : \mathbb{R} \to [0,1]$ of class C^∞, such that

$$\chi_n(Y) = \begin{cases} 1, & \text{if } |Y| \leq n, \\ 0, & \text{if } |Y| > n+1. \end{cases}$$

Consider the classical Galerkin approximation scheme defined by the processes $Y_n(t) \in P_nH$, solutions of

$$\begin{cases} dY_n + AY_n dt + P_nB_n(Y_n,Y_n)dt + P_nG(Y_n)dt = P_n\Psi(Y_n)dW(t), \quad t \in [0,T] \\ Y_n(0) = P_ny_0. \end{cases} \tag{4.33}$$

Noticing (2.29), (2.30) and B is locally Lipschitz from $V \times V$ to $D(A^{-\frac{3}{2}})$, all the coefficients are continuous and with linear growth in P_nH, thus, this equation has a unique weak solution $Y_n \in L^2(\Omega; C(0,T; P_nH))$. Moreover, for each $p \geq 2$, applying Itô formula on $|Y_n|^p$ and using the classical method referred to [7], by

$$\begin{aligned} \Big|(Y_n, G(Y_n))\Big| &\leq \frac{2}{\sqrt{C_0}}|S_n|\|v_n\| \\ &\leq \frac{1}{2}\|Y_n\|^2 + \frac{2}{C_0}|Y_n|^2, \end{aligned}$$

we can prove that there exist two positive constants $C_1(p)$, C_2, which are independent of n, such that

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_n(s)|^p\right) \leq C_1(p), \tag{4.34}$$

$$\mathbb{E}\int_0^T \|Y_n(s)\|^2 ds \leq C_2. \tag{4.35}$$

Step 2. Decompose Y_n as

$$\begin{aligned} Y_n(t) &= P_n y_0 - \int_0^t AY_n(s)ds - \int_0^t P_n B_n(Y_n(s), Y_n(s))ds \\ &\quad - \int_0^t P_n G(Y_n(s))ds + \int_0^t P_n \Psi(Y_n(s))dW(s) \\ &= J_n^1 + J_n^2(t) + J_n^3(t) + J_n^4(t) + J_n^5(t). \end{aligned}$$

By (4.34),

$$\mathbb{E}|J_n^1|^2 \leq C_3, \quad \mathbb{E}|J_n^4|^2_{W^{1,2}(0,T;V')} \leq C\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_n(s)|^2\right) \leq C_4.$$

From (4.35),

$$\mathbb{E}|J_n^2|^2_{W^{1,2}(0,T;V')} \leq C\mathbb{E}\int_0^T \|Y_n(s)\|^2 ds \leq C_5.$$

for suitable positive constants C_3, C_4, C_5. As to J_n^5, from Lemma 4.1 and (4.34), under **Hypothesis H0**, we have

$$\mathbb{E}|J_n^5|^2_{W^{\beta,2}(0,T;H)} \leq C_6(\beta),$$

for all $\beta \in (0, \frac{1}{2})$, and for some constant $C_6(\beta) > 0$. As to J_n^3, refer to [19],

$$\|B(Y, Y_1)\|_{-3} \leq C|Y|\|Y_1\|,$$

then

$$|P_n B_n(Y_n, Y_n)|^2_{L^2(0,T;D(A^{-\frac{3}{2}}))} \leq C_7 \sup_{0\leq s\leq T}|Y_n(s)|^2 \int_0^T \|Y_n(s)\|^2 ds,$$

by (4.34) and (4.35),

$$\mathbb{E}|J_n^3|_{W^{1,2}(0,T;D(A^{-\frac{3}{2}}))} \leq C_8\sqrt{C_1(2)C_2}.$$

For a Banach space B, we know $W^{1,p}(0,T;B) \subset W^{\alpha,p}(0,T;B)$ for all $\alpha \in (0,1)$ and $p > 1$. Collecting all the previous inequalities we obtain

$$\mathbb{E}|Y_n|_{W^{\beta,2}(0,T;D(A^{-\frac{3}{2}}))} \leq C_9(\beta),$$

for all $\beta \in (0, \frac{1}{2})$, and for some constant $C_9(\beta) > 0$. Recalling (4.35), which implies that the laws $\mathcal{D}(Y_n)$ are bounded in probability in

$$L^2(0,T;V) \cap W^{\beta,2}\left(0,T;D(A^{-\frac{3}{2}})\right),$$

thus, by Lemma 4.2, we have the family $\mathcal{D}(Y_n)$ is tight in $L^2(0,T;H)$.

Arguing similarly on the term J_n^5, on the basis of the estimate (4.34), we apply Lemma 4.3 and have that the family $\mathcal{D}(Y_n)$ is tight in $C\left(0,T;D(A^{-\frac{\gamma}{2}})\right)$, for all given $\gamma > 3$. Thus we can find a subsequence, still denoted by Y_n, such that $\mathcal{D}(Y_n)$ converges weakly in $L^2(0,T;H) \cap C\left(0,T;D(A^{-\frac{\gamma}{2}})\right)$.

Step 3. Fix $\gamma > 3$. By Skorohod embedding theorem, there exists a stochastic basis $(\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t\in[0,T]}, P^1)$ and $L^2(0,T;H) \cap C\left(0,T;D(A^{-\frac{\gamma}{2}})\right)$-valued random variables $Y^1, Y_n^1, n \geq 1$, such that Y_n^1 has the same law of Y_n on $L^2(0,T;H) \cap C\left(0,T;D(A^{-\frac{\gamma}{2}})\right)$, and $Y_n^1 \to Y^1$ in $L^2(0,T;H) \cap C\left(0,T;D(A^{-\frac{\gamma}{2}})\right)$, $P^1 - a.s.$ Of course, for each n,

$$\mathcal{D}(Y_n^1)(C(0,T;P_nH)) = 1,$$

and by (4.34) and (4.35), we have

$$\begin{aligned}
\mathbb{E}^{P^1}\left(\sup_{0\leq s\leq T}|Y_n^1(s)|^p\right) &\leq C_1(p), \\
\mathbb{E}^{P^1}\int_0^T \|Y_n^1(s)\|^2 ds &\leq C_2,
\end{aligned}$$

for all n and $p \geq 2$. Hence, we also have

$$Y^1 \in L^2(0,T;V) \cap L^\infty(0,T;H) \quad P^1 - a.s.$$

and $Y_n^1 \to Y^1$ weakly in $L^2(\Omega \times [0,T];V)$.

For each $n \geq 1$, the process $M_n^1(t)$ with trajectories in $C(0,T;H)$ defined as

$$M_n^1(t) = Y_n^1(t) - P_nY^1(0) + \int_0^t AY_n^1(s)ds + \int_0^t P_nB_n(Y_n^1(s), Y_n^1(s))ds + \int_0^t P_nG(Y_n^1(s))ds.$$

In fact, M_n^1 is a square integrable martingale with respect to the filtration

$$\left\{\mathcal{G}_n^1\right\}_t = \sigma\left\{Y_n^1(s), s \leq t\right\},$$

with quadratic variation

$$[M_n^1]_t = \int_0^t P_n\Psi(Y_n^1)\Psi(Y_n^1)^* P_n ds.$$

Then by a standard method (cf. [7]), we obtain the weak solution. ■

5 Exponential Convergence

Let λ and Y be a probability measure and a random variable on $(\Omega, \mathcal{F})$, respectively. The distribution $\mathcal{D}_\lambda(Y)$ denotes the law of Y under λ. A weak solution $\mathbb{P}_\mu$ with initial law μ is said to be stationary if, for any $t \geq 0$, μ is equal to $\mathcal{D}_{\mathbb{P}_\mu}(Y(t))$.

For $N \in \mathbb{N}$, we consider the following finite dimensional approximation of (2.31)

$$\begin{cases} dY_N + AY_N dt + P_N B(Y_N, Y_N)dt + P_N G(Y_N)dt = P_N \Psi(Y_N)dW(t), \\ P_N Y(0) = P_N y_0. \end{cases} \tag{5.36}$$

It's easy to show that for any given $y_0 \in H$, (5.36) has a unique solution $Y_N = Y_N(\cdot, y_0) = (v_N(\cdot, y_0), S_N(\cdot, y_0))$. Define

$$(\mathcal{P}_t^N \psi)(y_0) = \mathbb{E}[\psi(Y_N(t, y_0))], \quad \text{for } \psi \in \mathrm{B_b(P_N H)}.$$

$Y_N(\cdot, y_0)$ verifies the strong Markov property, which obviously implies that $(\mathcal{P}_t^N)_{t \in \mathbb{R}^+}$ is a Markov transition semi-group on $P_N H$. **In the following, P_N is omitted for simplicity written.**

Itô formula on $|Y_N(\cdot, y_0)|^2$ gives

$$\begin{aligned} d|Y_N|^2 + 2\|Y_N\|^2 dt = &-2\Big(Y_N, B(Y_N, Y_N)\Big)dt - 2\Big(Y_N, G(Y_N)\Big)dt \\ &+ 2\Big(Y_N, \Psi(Y_N)dW\Big) + \|\Psi(Y_N)\|^2_{\mathcal{L}_2(H,H)} dt. \end{aligned}$$

By integration by parts, we have

$$\Big(Y_N, B(Y_N, Y_N)\Big) = 0,$$

By Lemma 2.3, we obtain

$$\begin{aligned} &d|Y_N|^2 + 2\|Y_N\|^2 dt \\ &= 2\Big(Y_N, \Psi(Y_N)dW\Big) + 2\left(v_N, \frac{1}{\sqrt{C_0}} \int_{-1}^{z} \nabla_H S_N dz'\right)dt + \|\Psi(Y_N)\|^2_{\mathcal{L}_2(H,H)} dt \\ &\leq 2\Big(Y_N, \Psi(Y_N)dW\Big) + \frac{2}{\sqrt{C_0}}|S_N|\|v_N\|dt + \|\Psi(Y_N)\|^2_{\mathcal{L}_2(H,H)} dt. \end{aligned}$$

Then, by $ab \leq \frac{1}{2}(a^2 + b^2)$ and $\|Y_N\|^2 = \|v_N\|^2 + \|S_N\|^2$, we have

$$d|Y_N|^2 + \|Y_N\|^2 dt + \|S_N\|^2 dt \leq 2(Y_N, \Psi(Y_N)dW) + \frac{8}{C_0}|S_N|^2 dt + \kappa dt.$$

Since $C_0 \geq \frac{8}{\lambda_1}$, $\|S_N\|^2 \geq \lambda_1 |S_N|^2$, then $\frac{8}{C_0}|S_N|^2 \leq \|S_N\|^2$. Thus

$$d|Y_N|^2 + \|Y_N\|^2 dt \leq 2(Y_N, \Psi(Y_N)dW) + \kappa dt. \tag{5.37}$$

By $\|Y_N\|^2 \geq \mu_1 |Y_N|^2$ and applying integration by parts to $e^{\mu_1 t}|Y_N(t)|^2$, then integrating and taking expectation, we obtain

$$\mathbb{E}\left(|Y_N(t)|^2\right) \leq e^{-\mu_1 t}|y_0|^2 + \frac{\kappa}{\mu_1}. \tag{5.38}$$

Hence, applying the Krylov-Bogoliubov Criterion (see [4]), we obtain that $(\mathcal{P}_t^N)_t$ admits an invariant measure μ_N and that every invariant measure has a moment of order two in H. Let Y_0^N be a random variable whose law is μ_N and which is independent of W, then $Y_N = Y_N(\cdot, Y_0^N)$ is a stationary solution of (5.36). Integrating (5.37), we obtain

$$\mathbb{E}|Y_N(t)|^2 + \mathbb{E}\int_0^t \|Y_N(s)\|^2 ds \leq \mathbb{E}|Y_N(0)|^2 + \kappa t. \tag{5.39}$$

Since the law of $Y_N(s)$ is μ_N for any $s \geq 0$, it follows

$$\int_{P_N H} \|y\|^2 \mu_N(dy) \leq \kappa. \tag{5.40}$$

Using the similar arguments as proof of Theorem 3.1, the laws $(\mathbb{P}_{\mu_N}^N)$ of $Y_N(\cdot, Y_0^N)$ are tight in $L^2(0,T;H) \cap C(0,T;D(A^{-\frac{\gamma}{2}}))$, $\gamma > 3$. Then, for a subsequence, still denote by $(\mathbb{P}_{\mu_N}^N)$, which converges in law to $\mathbb{P}_\mu$ a stationary martingale solution of (2.31) with initial law μ. We deduce from (5.40) that

$$\int_H \|y\|^2 \mu(dy) \leq \kappa.$$

In general, it's not known whether μ is an invariant measure due to the lack of uniqueness and also we don't know whether (2.31) defines a Markov evolution. However, the above information is a key to prove uniqueness of invariant measures.

5.1 Coupling Method

Let (λ_1, λ_2) be two distributions on a Polish space $(E, \mathcal{B}(E))$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (Z_1, Z_2) be two random variables $(\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$. We say that (Z_1, Z_2) is a coupling of (λ_1, λ_2) if $\lambda_i = \mathcal{D}(Z_i)$ for $i = 1, 2$. The total variation $\|\lambda\|_{var}$ of a finite real measure λ on E is defined as

$$\|\lambda\|_{var} = \sup\left\{|\lambda(\Gamma)| \mid \Gamma \in \mathcal{B}(E)\right\},$$

where $\mathcal{B}(E)$ stands for the set of the Borelian subsets of E.

The next result is fundamental in the coupling methods, the proof, for example, can be found in [26].

Lemma 5.1. *Let (λ_1, λ_2) be two probability measures on $(E, \mathcal{B}(E))$. Then*

$$\|\lambda_1 - \lambda_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

The minimum is taken over all couplings (Z_1, Z_2) of (λ_1, λ_2). There exists a coupling which reaches the minimum value. It is called a maximal coupling.

Let us first consider the case of the solutions of (5.36). Assume that **Hypothesis H1** holds. Let $N \in \mathbb{N}$ and $(y_0^1, y_0^2) \in H \times H$. Combining arguments similar as [15],[22], it can be shown that there exists a function $p_N(\cdot) > 0$ such that

$$\|(\mathcal{P}_1^N)^* \delta_{y_0^2} - (\mathcal{P}_1^N)^* \delta_{y_0^1}\|_{var} \leq 1 - p_N(|y_0^1| + |y_0^2|), \tag{5.41}$$

where $\delta_{y_0^1}$ and $\delta_{y_0^2}$ are two Dirac measures on single point y_0^1 and y_0^2, respectively. Applying Lemma 5.1, we construct a maximal coupling $(Z_1, Z_2) = (Z_1(y_0^1, y_0^2), Z_2(y_0^1, y_0^2))$ of $\left((\mathcal{P}_1^N)^*\delta_{y_0^1}, (\mathcal{P}_1^N)^*\delta_{y_0^2}\right)$. It follows

$$\mathbb{P}(Z_1 = Z_2) \geq p_N(|y_0^1| + |y_0^2|) > 0. \tag{5.42}$$

Let $(W, \tilde{W})$ be a couple of independent cylindrical Wiener processes on $H \times H$ and $\delta > 0$. Denote by $Y_N(\cdot, y_0^1)$ and $\tilde{Y}_N(\cdot, y_0^2)$ the solutions of (5.36) with initial data y_0^1 and y_0^2 associated to W and $\tilde{W}$, respectively. Now we can construct a couple of random variables $(V_1, V_2) = (V_1(y_0^1, y_0^2), V_2(y_0^1, y_0^2))$ on $P_N H$ for $(Y_N, \tilde{Y}_N)$ as follows

$$(V_1, V_2) = \begin{cases} (Y_N(\cdot, y_0), Y_N(\cdot, y_0)), & \text{if } y_0^1 = y_0^2 = y_0, \\ (Z_1(y_0^1, y_0^2), Z_2(y_0^1, y_0^2)), & \text{if } (y_0^1, y_0^2) \in B_{H\times H}(0, \delta)\backslash\{y_0^1 = y_0^2\}, \\ (Y_N(\cdot, y_0^1), \tilde{Y}_N(\cdot, y_0^2)), & \text{else,} \end{cases} \tag{5.43}$$

where $B_{H\times H}(0, \delta)$ is the ball of $H \times H$ with radius δ . Then $(V_1(y_0^1, y_0^2), V_2(y_0^1, y_0^2))$ is a coupling of $\left((\mathcal{P}_1^N)^*\delta_{y_0^1}, (\mathcal{P}_1^N)^*\delta_{y_0^2}\right)$. It can be shown that it depends on (y_0^1, y_0^2). We then construct a coupling (Y^1, Y^2) of $\left(\mathcal{D}(Y_N(\cdot, y_0^1)), \mathcal{D}(Y_N(\cdot, y_0^2))\right)$ by induction on $\mathbb{N}$. Firstly, setting $Y^i(0) = y_0^i$ for $i = 1, 2$, then assuming that we have constructed (Y^1, Y^2) on $\{0, 1, \cdots, k\}$. We take (V_1, V_2) as above independent of (Y^1, Y^2) and set

$$Y^i(k + 1) = V_i(Y^1(k), Y^2(k)) \quad \text{for i} = 1, 2.$$

Taking into account (5.38), it is easily shown that the time of return of (Y^1, Y^2) in $B_{H\times H}(0, \frac{4\kappa}{\mu_1})$ admits an exponential moment. We choose $\delta = \frac{4\kappa}{\mu_1}$. It follows from (5.42) and (5.43) that $(Y^1(n), Y^2(n)) \in B_{H\times H}(0, \delta)$ implies that the probability of (Y^1, Y^2) having coupled at time $n + 1$ is bounded below by $p_N(2\delta) > 0$. Finally, remark that if (Y^1, Y^2) are coupled at time $n + 1$, then they remain coupled for any time after. Combining these properties and using the fact that $(Y^1(n), Y^2(n))_{n\in\mathbb{N}}$ is a discrete strong Markov process, it is easily shown that

$$\mathbb{P}(Y^1(n) \neq Y^2(n)) \leq C_N e^{-\gamma_N n}(1 + |y_0^1|^2 + |y_0^2|^2), \tag{5.44}$$

with $\gamma_N > 0$. Recall that (Y^1, Y^2) is a coupling of $\left(\mathcal{D}(Y_N(\cdot, y_0^1)), \mathcal{D}(Y_N(\cdot, y_0^2))\right)$ on $\mathbb{N}$. It follows that $(Y^1(n), Y^2(n))$ is a coupling of $\left((\mathcal{P}_n^N)^*\delta_{y_0^1}, (\mathcal{P}_n^N)^*\delta_{y_0^2}\right)$. Combining Lemma 5.1 and (5.44), we obtain, for $n \in \mathbb{N}$,

$$||(\mathcal{P}_n^N)^*\delta_{y_0^2} - (\mathcal{P}_n^N)^*\delta_{y_0^1}||_{var} \leq C_N e^{-\gamma_N n}(1 + |y_0^1|^2 + |y_0^2|^2).$$

Setting $n = \lfloor t \rfloor$(the integer part of t) and integrating (y_0^2, y_0^1) over $((\mathcal{P}_{t-n}^N)^*\lambda) \bigotimes \mu_N$, where μ_N is an invariant measure, it follows that, for any $\lambda \in \mathcal{P}(P_N H)$ with $\int_{P_N H} |y|^2 \lambda(dy) < \infty$,

$$||(\mathcal{P}_t^N)^*\lambda - \mu_N||_{var} \leq C_N e^{-\gamma_N t}\left(1 + \int_{P_N H} |y|^2 \lambda(dy)\right). \tag{5.45}$$

This result is useless when considering equations (2.31) since the constants C_N, γ_N strongly depend on N. In the following, we prove that (5.41) is true uniformly in N, which implies (5.45) holds with

constants uniform in N, provided y_0^1, y_0^2 in a small ball of $\mathbb{H}_3$. Then it remains to prove that the time of return in this small ball admits an exponential moment. Besides, in order to obtain Theorem 3.2, it's sufficient to prove the following proposition.

Proposition 5.1. *Assume that **Hypothesis H1** holds. Then there exist $C = C(\Psi, \mathbb{T}^3, \varepsilon_0) > 0$ and $\gamma = \gamma(\Psi, \mathbb{T}^3, \varepsilon_0) > 0$ such that for any $N \in \mathbb{N}$, there exists a unique invariant measure μ_N for $(\mathcal{P}_t^N)_{t\in\mathbb{R}^+}$. Moreover, for any $\lambda \in \mathcal{P}(P_N H)$ with $\int_{P_N H} |y|^2 \lambda(dy) < \infty$,*

$$\|(\mathcal{P}_t^N)^*\lambda - \mu_N\|_{var} \le Ce^{-\gamma t}\left(1 + \int_{P_N H} |y|^2 \lambda(dy)\right). \tag{5.46}$$

Let $\lambda \in \mathcal{P}(H)$ and $\mathbb{E}_\lambda$ be an expectation under the initial distribution λ. It's easy to know that (5.46) implies that

$$\left|\mathbb{E}_\lambda\Big(g(Y_N(t))\Big) - \int_{P_N H} g(y)\mu_N(dy)\right| \le Ce^{-\gamma t}|g|_\infty \left(1 + \int_H |y|^2 \lambda(dy)\right), \tag{5.47}$$

for any $g \in UC_b(\mathbb{H}_s)$.

5.2 Coupling of Solutions Starting from Small Initial Data

The aim of this section is to establish the following result, which is analogous to (5.42) but uniform in N.

Proposition 5.2. *Assume that **Hypothesis H1** holds. Then there exist $(\Upsilon, \delta) \in (0,1) \times (0,1)$ such that, for any $N \in \mathbb{N}$, there exists a coupling $(Z_1(y_0^1, y_0^2), Z_2(y_0^1, y_0^2))$ of $\left((\mathcal{P}_\Upsilon^N)^*\delta_{y_0^1}, (\mathcal{P}_\Upsilon^N)^*\delta_{y_0^2}\right)$ which measurably depends on $(y_0^1, y_0^2) \in \mathbb{H}_3 \times \mathbb{H}_3$ and verifies*

$$\mathbb{P}\left(Z_1(y_0^1, y_0^2) = Z_2(y_0^1, y_0^2)\right) \ge \frac{3}{4}, \tag{5.48}$$

provided

$$\|y_0^1\|_3^2 \vee \|y_0^2\|_3^2 \le \delta. \tag{5.49}$$

Assume that **Hypothesis H1** holds. Let $\Upsilon \in (0,1)$. Applying Lemma 5.1, we can construct $\left(Z_1(y_0^1, y_0^2), Z_2(y_0^1, y_0^2)\right)$ as the maximal coupling of $(\mathcal{P}_\Upsilon^*\delta_{y_0^1}, \mathcal{P}_\Upsilon^*\delta_{y_0^2})$. Measurable dependence on (y_0^1, y_0^2) follows from a slight extension of Lemma 5.1 (cf. [27], Remark A.1). In order to establish Proposition 5.2, it is sufficient to prove that there exists $c(\kappa, \mathbb{T}^3)$ not depending on $\Upsilon \in (0,1)$ and on $N \in \mathbb{N}$ such that

$$\|(\mathcal{P}_\Upsilon^N)^*\delta_{y_0^2} - (\mathcal{P}_\Upsilon^N)^*\delta_{y_0^1}\|_{var} \le c(\kappa, \mathbb{T}^3)\sqrt{\Upsilon}, \tag{5.50}$$

provided

$$\|y_0^1\|_3^2 \vee \|y_0^2\|_3^2 \le \kappa\Upsilon^3. \tag{5.51}$$

Then it suffices to choose $\Upsilon \le 1/(4c(\kappa, \mathbb{T}^3))^2$ and $\delta = \kappa\Upsilon^3$.

Since $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$, (5.50) is equivalent to

$$\left|\mathbb{E}\left(g(Y_N(\Upsilon, y_0^2)) - g(Y_N(\Upsilon, y_0^1))\right)\right| \leq 8|g|_\infty c(\kappa, \mathbb{T}^3)\sqrt{\Upsilon}, \tag{5.52}$$

for any $g \in UC_b(P_N H)$. It follows from the density of $C_b^1(P_N H) \subset UC_b(P_N H)$ that, in order to establish Proposition 5.2, it is sufficient to prove that (5.52) holds for any $N \in \mathbb{N}, \Upsilon \in (0,1)$ and $g \in C_b^1(P_N H)$ provided (5.51) holds.

The proof of (5.52) under condition (5.51) is aligned into the next three subsections.

5.2.1 A Prior Estimate.

For any process $Y = (v, S)$, we define the $\mathbb{H}_2-$ energy of Y at time t by

$$E_Y^{\mathbb{H}_2}(t) \triangleq \|Y(t)\|_2^2 + \int_0^t \|Y(s)\|_3^2 ds.$$

For any $N \in \mathbb{N}$. Let P_N^1 be the orthogonal projection in H_1 onto the space $P_N^1 H_1 \triangleq \mathrm{Span}\{k_1, k_2, \cdots, k_N\}$ and P_N^2 be the orthogonal projection in H_2 onto the space $P_N^2 H_2 \triangleq \mathrm{Span}\{l_1, l_2, \cdots, l_N\}$. Denote $P_N Y = (P_N^1 v, P_N^2 S)$ for any $Y = (v, S) \in V$. We obtain the following finite dimension approximation of (2.23), (2.24)

$$\begin{cases} \dfrac{\partial v_N}{\partial t} + P_N^1(v_N \cdot \nabla_H)v_N + P_N^1\Phi(v_N)\dfrac{\partial v_N}{\partial z} + P_N^1 fk \times v_N + P_N^1 \nabla_H p_b \\ \qquad - \dfrac{1}{\sqrt{C_0}} P_N^1 \displaystyle\int_{-1}^{z} \nabla_H S_N dz' - \Delta v_N = P_N^1 \phi(v_N, S_N)\dfrac{dW_1}{dt}, \\ v_N(0) = P_N^1 v_0, \end{cases} \tag{5.53}$$

$$\begin{cases} \dfrac{\partial S_N}{\partial t} + P_N^2(v_N \cdot \nabla_H)S_N + P_N^2\Phi(v_N)\dfrac{\partial S_N}{\partial z} - \Delta S_N = P_N^2 \varphi(v_N, S_N)\dfrac{dW_2}{dt}, \\ S_N(0) = P_N^2 S_0. \end{cases} \tag{5.54}$$

For written simplicity, in the following, P_N, P_N^1 and P_N^2 are omitted. Now we establish the following result which will be useful in the proof of (5.52).

Lemma 5.2. *Assume that **Hypothesis H1** holds. There exist $K_0 = K_0(\mathbb{T}^3)$ and $c = c(\mathbb{T}^3)$ such that for any $\Upsilon \leq 1$ and any $N \in \mathbb{N}$, we have*

$$\mathbb{P}\left(\sup_{(0,\Upsilon)} E_{Y_N(\cdot, y_0)}^{\mathbb{H}_2}(t) > K_0\right) \leq c\left(1 + \frac{\kappa}{K_0}\right)\sqrt{\Upsilon},$$

provided $\|y_0\|_2^2 \leq \kappa\Upsilon$.

Proof of Lemma 5.2. Let $Y_N = Y_N(\cdot, y_0)$, $v_N = v_N(\cdot, y_0)$ and $S_N = S_N(\cdot, y_0)$. Itô formula on $|\Delta v_N|^2$ gives

$$d|\Delta v_N|^2 + 2|\nabla \Delta v_N|^2 dt = dM_{\mathbb{H}_2}^v + I_{\mathbb{H}_2}^v dt + J_{\mathbb{H}_2}^v dt + \|P_N^1 \phi(Y_N)\|_{\mathcal{L}_2(H;\mathbb{H}_2)}^2 dt,$$

where

$$\begin{cases} I^v_{\mathbb{H}_2} = -2\left(\Delta v_N, \Delta[(v_N \cdot \nabla_H)v_N + \Phi(v_N)\dfrac{\partial v_N}{\partial z}]\right), \\ J^v_{\mathbb{H}_2} = -2\left(\Delta v_N, \Delta[fk \times v_N + \nabla_H p_b - \dfrac{1}{\sqrt{C_0}}\displaystyle\int_{-1}^{z} \nabla_H S_N dz']\right), \\ M^v_{\mathbb{H}_2} = 2\displaystyle\int_0^t (\Delta Y_N(s), \Delta\phi(Y_N(s))dW(s)). \end{cases}$$

By Lemma 2.4, Hölder inequality and the Young inequality, we obtain

$$\begin{aligned} I^v_{\mathbb{H}_2} &= 2\left(\nabla\Delta v_N, \nabla[(v_N \cdot \nabla_H)v_N + \Phi(v_N)\frac{\partial v_N}{\partial z}]\right) \\ &\le c|\nabla\Delta v_N||\nabla v_N|^{\frac{1}{2}}|\Delta v_N|^{\frac{1}{2}}|\Delta v_N| + c|\nabla\Delta v_N||\Delta v_N|^2 \\ &\quad + c|\nabla\Delta v_N||\Delta v_N|^{\frac{1}{2}}|\nabla\Delta v_N|^{\frac{1}{2}}|\nabla v_N|^{\frac{1}{2}}|\Delta v_N|^{\frac{1}{2}} + c|\nabla\Delta v_N||\nabla v_N|^{\frac{1}{2}}|\Delta v_N||\nabla\Delta v_N|^{\frac{1}{2}} \\ &\le \frac{1}{4}|\nabla\Delta v_N|^2 + c|\Delta v_N|_2^4 + c|\Delta v_N|_2^6, \\ J^v_{\mathbb{H}_2} &= 2\left(\nabla\Delta v_N, \nabla[fk \times v_N + \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z} \nabla_H S_N dz']\right) \\ &\le \frac{1}{4}|\nabla\Delta v_N|^2 + c|\Delta v_N|^2 + c|\Delta S_N|_2^2. \end{aligned}$$

Similarly, Itô formula on $|\Delta S_N|^2$ gives

$$d|\Delta S_N|^2 + 2|\nabla\Delta S_N|^2 dt = dM^S_{\mathbb{H}_2} + I^S_{\mathbb{H}_2}dt + J^S_{\mathbb{H}_2}dt + \|P_N^2\varphi(Y_N)\|^2_{\mathcal{L}_2(H;\mathbb{H}_2)}dt,$$

where

$$\begin{cases} I^S_{\mathbb{H}_2} = -2\left(\Delta S_N, \Delta[(v_N \cdot \nabla_H)S_N + \Phi(v_N)\dfrac{\partial S_N}{\partial z}]\right), \\ J^S_{\mathbb{H}_2} = -2(\Delta S_N, \Delta\phi(v_N)), \\ M^S_{\mathbb{H}_2} = 2\displaystyle\int_0^t (\Delta Y_N(s), \Delta\varphi(Y_N(s))dW(s)). \end{cases}$$

Using the same method as above, we have

$$\begin{aligned} I^S_{\mathbb{H}_2} &= 2\left(\nabla\Delta S_N, \nabla[(v_N \cdot \nabla_H)S_N + \Phi(v_N)\frac{\partial S_N}{\partial z}]\right) \\ &\le c|\nabla\Delta S_N||\nabla v_N|^{\frac{1}{2}}|\Delta v_N|^{\frac{1}{2}}|\Delta S_N| + c|\nabla\Delta S_N||\Delta v_N||\Delta S_N| \\ &\quad + c|\nabla\Delta S_N||\nabla\Delta v_N|^{\frac{1}{2}}|\Delta v_N|^{\frac{1}{2}}|\Delta S_N| + c|\nabla\Delta S_N||\Delta v_N||\nabla\Delta S_N|^{\frac{1}{2}}|\Delta S_N|^{\frac{1}{2}} \\ &\le \frac{1}{4}|\nabla\Delta S_N|^2 + c|\Delta v_N|^4 + c|\Delta v_N|^8 + c|\Delta S_N|^8 + c|\Delta S_N|^4, \\ J^S_{\mathbb{H}_2} &= 2(\nabla\Delta S_N, \nabla\Phi(v_N)) \\ &\le \frac{1}{4}|\nabla\Delta S_N|^2 + c|\Delta v_N|^2. \end{aligned}$$

Combining the above inequalities, we obtain

$$d\|Y_N\|_2^2 + \frac{3}{2}\|Y_N\|_3^2 dt \le c\|Y_N\|_2^8 dt + c\kappa dt + dM_{\mathbb{H}_2},$$

then

$$d\|Y_N\|_2^2 + \|Y_N\|_3^2 dt \le c\|Y_N\|_2^2 \left(\|Y_N\|_2^6 - 8K_0^3\right) dt + c\kappa dt + dM_{\mathbb{H}_2},$$

where

$$K_0 = \sqrt[3]{\frac{\mu_1}{16c}}. \tag{5.55}$$

Setting $\sigma_{\mathbb{H}_2} = \inf\{t \in (0,\Upsilon) \mid \|Y_N(t)\|_2^2 > 2K_0\}$, we infer from $\|y_0\|_2^2 \le \kappa\Upsilon$ that for any $t \in (0, \sigma_{\mathbb{H}_2})$

$$\mathbb{E}_{Y_N}^{\mathbb{H}_2}(t) \le M_{\mathbb{H}_2}(t) + c\kappa\Upsilon. \tag{5.56}$$

We deduce from **Hypothesis H1** that $\Psi(y)^* A$ is bounded in $\mathcal{L}(\mathbb{H}_2; \mathbb{H}_2)$ by $c\kappa$. It follows that for any $t \in (0, \sigma_{\mathbb{H}_2})$,

$$\langle M_{\mathbb{H}_2}\rangle(t) = 4\int_0^t |P_N\Psi(Y_N(s))^* A^2 Y_N(s)|^2 ds \le c\kappa \int_0^t \|Y_N\|_2^2 ds \le 2c\kappa K_0\Upsilon.$$

Hence Burkholder-Davis-Gundy inequality gives

$$\mathbb{E}\left(\sup_{(0,\sigma_{\mathbb{H}_2})} M_{\mathbb{H}_2}(t)\right) \le c\mathbb{E}\sqrt{\langle M_{\mathbb{H}_2}\rangle(\sigma_{\mathbb{H}_2})} \le c\sqrt{K_0\kappa\Upsilon} \le c(K_0+\kappa)\sqrt{\Upsilon}.$$

It follows from (5.56) and $\Upsilon \le 1$ that

$$\mathbb{E}\left(\sup_{(0,\sigma_{\mathbb{H}_2})} E_{Y_N}^{\mathbb{H}_2}(t)\right) \le c(K_0+\kappa)\sqrt{\Upsilon},$$

which yields, by Chebyshev inequality,

$$\mathbb{P}\left(\sup_{(0,\sigma_{\mathbb{H}_2})} E_{Y_N}^{\mathbb{H}_2}(t) > K_0\right) \le c(1+\frac{\kappa}{K_0})\sqrt{\Upsilon}.$$

Let $B = \left\{\sup_{(0,\sigma_{\mathbb{H}_2})} E_{Y_N}^{\mathbb{H}_2}(t) \le K_0\right\}$, $A = \left\{\sup_{(0,\Upsilon)} E_{Y_N}^{\mathbb{H}_2}(t) \le K_0\right\}$. Since $\sup_{(0,\sigma_{\mathbb{H}_2})} E_{Y_N}^{\mathbb{H}_2}(t) \le K_0$ implies $\sigma_{\mathbb{H}_2} = \Upsilon$, we have $B \subset A$, i.e. $B^c \supset A^c$, then $\mathbb{P}(A^c) \le \mathbb{P}(B^c)$, which implies Lemma 5.2. ■

5.2.2 Estimate of Derivative

Let $N \in \mathbb{N}$ and $(y_0, h) \in (\mathbb{H}_3)^2$, where $y_0 = (v_0, S_0)$. Consider

$$\begin{cases} \dfrac{\partial\beta_N}{\partial t} + P_N(v_N\cdot\nabla_H)\beta_N + P_N\Phi(v_N)\dfrac{\partial\beta_N}{\partial z} + P_N(\eta_N\cdot\nabla_H)Y_N + P_N\Phi(\eta_N)\dfrac{\partial Y_N}{\partial z} \\ \qquad\qquad + P_N\tilde{G}(\beta_N) + A\beta_N = P_N\Psi'(Y_N)\beta_N\dfrac{dW}{dt}, \\ \beta_N(s,s,y_0)\cdot h = P_N h, \end{cases} \tag{5.57}$$

where

$$\tilde{G}(\beta_N) = \begin{pmatrix} P_N^1 fk\times\eta_N - \frac{1}{\sqrt{C_0}}P_N^1\int_{-1}^z \nabla_H\gamma_N dz' \\ 0 \end{pmatrix},$$

and

$$\eta_N(t) = \eta_N(t,s,y_0)\cdot h, \quad \gamma_N(t) = \gamma_N(t,s,y_0)\cdot h \quad \text{for } t \geq s.$$

Denote $\beta_N = (\eta_N, \gamma_N)$ and $\beta_N(t) = \beta_N(t,s,y_0)\cdot h$. Existence and uniqueness of the solutions of (5.57) is easily shown. Moreover, if $g \in C_b^1(P_N H)$, then, for any $t \geq 0$, we have

$$\left(\nabla\left(\mathcal{P}_t^N g\right)(y_0), h\right) = \mathbb{E}\left(\nabla g(Y_N(t,y_0)), \beta_N(t,0,y_0)\cdot h\right).$$

For process $Y = (v, S)$, set

$$\sigma(Y) = \inf\left\{t \in (0,\Upsilon) | \int_0^t \|Y(s)\|_3^2 ds \geq K_0 + 1\right\}, \tag{5.58}$$

where K_0 is defined in (5.55).

Lemma 5.3. *Assume that **Hypothesis H1** holds. Then there exists $c = c(\kappa, \mathbb{T}^3)$ such that for any $N \in \mathbb{N}, \Upsilon \leq 1$ and $(y_0, h) \in (\mathbb{H}_3)^2$,*

$$\mathbb{E}\int_0^{\sigma(Y_N(\cdot,y_0))} \|\beta_N(t,0,y_0)\cdot h\|_4^2 dt \leq c\|h\|_3^2.$$

Proof of Lemma 5.3. For written simply, we set $\beta_N(t) = \beta_N(t,0,y_0)\cdot h$, $\eta_N(t) = \eta_N(t,0,y_0)\cdot h$, $\gamma_N(t) = \gamma_N(t,0,y_0)\cdot h$ and $\sigma = \sigma(Y_N(\cdot,y_0))$. Itô formula on $|\nabla\Delta\eta_N(t)|^2$ gives

$$d|\nabla\Delta\eta_N(t)|^2 + 2|\Delta^2\eta_N(t)|^2 dt = dM_{\eta_N} + I_{\eta_N}dt + J_{\eta_N}dt + \|P_N^1(\phi'(Y_N)\cdot\beta_N)\|^2_{\mathcal{L}_2(H;\mathbb{H}_3)}dt,$$

where

$$\begin{cases} M_{\eta_N} = 2\displaystyle\int_0^t (\nabla\Delta\eta_N, \nabla\Delta(P_N^1\phi'(Y_N)\cdot\beta_N)dW_1), \\ I_{\eta_N} = -2\left(\nabla\Delta\eta_N, \nabla\Delta[(v_N\cdot\nabla_H)\eta_N + (\eta_N\cdot\nabla_H)v_N]\right) - 2\left(\nabla\Delta\eta_N, \nabla\Delta[\Phi(v_N)\dfrac{\partial\eta_N}{\partial z} + \Phi(\eta_N)\dfrac{\partial v_N}{\partial z}]\right) = I^{(1)}_{\eta_N} + I^{(2)}_{\eta_N}, \\ J_{\eta_N} = -2\left(\nabla\Delta\eta_N, \nabla\Delta[fk\times\eta_N - \dfrac{1}{\sqrt{C_0}}\displaystyle\int_{-1}^z \nabla_H\gamma_N dz']\right). \end{cases}$$

By Lemma 2.4, Hölder inequality and the Young inequality, we obtain

$$
\begin{aligned}
I^{(1)}_{\eta_N} &= 2\left(\Delta^2\eta_N, \Delta[(v_N\cdot\nabla_H)\eta_N + (\eta_N\cdot\nabla_H)v_N]\right)\\
&\leq c|\Delta^2\eta_N||\Delta v_N|^{\frac{1}{2}}|\nabla\Delta v_N|^{\frac{1}{2}}|\Delta\eta_N| + c|\Delta^2\eta_N||\Delta\eta_N|^{\frac{1}{2}}|\nabla\Delta\eta_N|^{\frac{1}{2}}|\Delta v_N| + c|\Delta^2\eta_N||\nabla\Delta v_N||\Delta v_N|\\
&\quad +c|\Delta^2\eta_N||\Delta\eta_N|^{\frac{1}{2}}|\nabla\Delta\eta_N|^{\frac{1}{2}}|\Delta v_N| + c|\Delta^2\eta_N||\Delta v_N|^{\frac{1}{2}}|\nabla\Delta v_N|^{\frac{1}{2}}|\Delta\eta_N| + c|\Delta^2\eta_N||\nabla\Delta v_N||\Delta\eta_N|\\
&\leq \frac{1}{8}|\Delta^2\eta_N|^2 + c|\nabla\Delta\eta_N|^2|\nabla\Delta v_N|^2,\\
I^{(2)}_{\eta_N} &= 2\left(\Delta^2\eta_N, \Delta[\Phi(v_N)\frac{\partial\eta_N}{\partial z} + \Phi(\eta_N)\frac{\partial v_N}{\partial z}]\right)\\
&\leq c|\Delta^2\eta_N||\nabla\Delta v_N||\nabla\Delta\eta_N| + c|\Delta^2\eta_N||\Delta v_N|^{\frac{1}{2}}|\nabla\Delta v_N|^{\frac{1}{2}}|\Delta\eta_N|^{\frac{1}{2}}|\nabla\Delta\eta_N|^{\frac{1}{2}}\\
&\quad +c|\Delta^2\eta_N||\nabla\Delta v_N||\nabla\Delta\eta_N| + c|\Delta^2\eta_N||\nabla\Delta v_N||\nabla\Delta\eta_N| + c|\Delta^2\eta_N||\nabla\Delta v_N||\nabla\Delta\eta_N|\\
&\leq \frac{1}{8}|\Delta^2\eta_N|^2 + c|\nabla\Delta\eta_N|^2|\nabla\Delta v_N|^2,\\
J_{\eta_N} &= 2\left(\Delta^2\eta_N, \Delta[fk\times\eta_N - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H\gamma_N dz']\right)\\
&\leq \frac{1}{4}|\Delta^2\eta_N|^2 + c|\Delta\eta_N|^2 + c|\nabla\Delta\gamma_N|^2.
\end{aligned}
$$

Itô formula on $|\nabla\Delta\gamma_N(t)|^2$ gives

$$
d|\nabla\Delta\gamma_N(t)|^2 + 2|\Delta^2\gamma_N(t)|^2dt = dM_{\gamma_N} + I_{\gamma_N}dt + J_{\gamma_N}dt + \|P_N^2(\varphi'(Y_N)\cdot\beta_N)\|^2_{\mathcal{L}_2(H;\mathbb{H}_3)}dt,
$$

where

$$
\begin{cases}
M_{\gamma_N}(t) = 2\int_0^t(\nabla\Delta\gamma_N, (\nabla\Delta P_N^2\varphi'(Y_N)\cdot\beta_N)dW_2),\\
I_{\gamma_N} = -2\Big(\nabla\Delta\gamma_N, \nabla\Delta[(v_N\cdot\nabla_H)\gamma_N + (\eta_N\cdot\nabla_H)S_N]\Big) - 2\Big(\nabla\Delta\gamma_N, \nabla\Delta[\Phi(v_N)\frac{\partial\gamma_N}{\partial z} + \Phi(\eta_N)\frac{\partial S_N}{\partial z}]\Big) = I^{(1)}_{\gamma_N} + I^{(2)}_{\gamma_N},\\
J_{\gamma_N} = -2\left(\nabla\Delta\gamma_N, \nabla\Delta\Phi(\eta_N)\right).
\end{cases}
$$

Using the same method as above, we have

$$
\begin{aligned}
I^{(1)}_{\gamma_N} &= 2\Big(\Delta^2\gamma_N, \Delta[(v_N\cdot\nabla_H)\gamma_N + (\eta_N\cdot\nabla_H)S_N]\Big)\\
&\leq c|\Delta^2\gamma_N||\Delta v_N|^{\frac{1}{2}}|\nabla\Delta v_N|^{\frac{1}{2}}|\Delta\gamma_N| + c|\Delta^2\gamma_N||\Delta\gamma_N|^{\frac{1}{2}}|\nabla\Delta\gamma_N|^{\frac{1}{2}}|\Delta v_N| + c|\Delta^2\gamma_N||\nabla\Delta\gamma_N||\Delta v_N|\\
&\quad +c|\Delta^2\gamma_N||\Delta\eta_N|^{\frac{1}{2}}|\nabla\Delta\eta_N|^{\frac{1}{2}}|\Delta S_N| + c|\Delta^2\gamma_N||\Delta S_N|^{\frac{1}{2}}|\nabla\Delta S_N|^{\frac{1}{2}}|\Delta\eta_N| + c|\Delta^2\gamma_N||\nabla\Delta S_N||\Delta\eta_N|\\
&\leq \frac{1}{8}|\Delta^2\gamma_N|^2 + c|\nabla\Delta v_N|^2|\nabla\Delta\gamma_N|^2 + c|\nabla\Delta\eta_N|^2|\nabla\Delta S_N|^2,\\
I^{(2)}_{\gamma_N} &= 2\Big(\Delta^2\gamma_N, \Delta[\Phi(v_N)\frac{\partial\gamma_N}{\partial z} + \Phi(\eta_N)\frac{\partial S_N}{\partial z}]\Big)\\
&\leq c|\Delta^2\gamma_N||\nabla\Delta v_N||\nabla\Delta\gamma_N| + c|\Delta^2\gamma_N||\Delta v_N|^{\frac{1}{2}}|\nabla\Delta v_N|^{\frac{1}{2}}|\Delta\gamma_N|^{\frac{1}{2}}|\nabla\Delta\gamma_N|^{\frac{1}{2}} + c|\Delta^2\gamma_N||\nabla\Delta\gamma_N||\nabla\Delta v_N|\\
&\quad +c|\Delta^2\gamma_N||\nabla\Delta S_N||\nabla\Delta\eta_N| + c|\Delta^2\gamma_N||\Delta S_N|^{\frac{1}{2}}|\nabla\Delta S_N|^{\frac{1}{2}}|\Delta\eta_N|^{\frac{1}{2}}|\nabla\Delta\eta_N|^{\frac{1}{2}} + c|\Delta^2\gamma_N||\nabla\Delta S_N||\nabla\Delta\eta_N|\\
&\leq \frac{1}{8}|\Delta^2\gamma_N|^2 + c|\nabla\Delta v_N|^2|\nabla\Delta\gamma_N|^2 + c|\nabla\Delta\eta_N|^2|\nabla\Delta S_N|^2,\\
J_{\gamma_N} &= 2\left(\Delta^2\gamma_N, \Delta\Phi(\eta_N)\right)\\
&\leq \frac{1}{2}|\Delta^2\gamma_N|^2 + c|\nabla\Delta\eta_N|^2.
\end{aligned}
$$

Collecting all inequalities in the above, we have

$$d\|\beta_N\|_3^2 + \frac{3}{2}\|\beta_N\|_4^2 dt \le c\|\beta_N\|_3^2(1 + \|Y_N\|_3^2)dt + dM_{\eta_N} + dM_{\gamma_N}. \tag{5.59}$$

Integrating and taking the expectation on both sides of (5.59), we deduce

$$\mathbb{E}\left(\mathcal{E}(\sigma,0)\|\beta_N(\sigma)\|_3^2 + \int_0^\sigma \mathcal{E}(t,0)\|\beta_N(t)\|_4^2 dt\right) \le \|h\|_3^2,$$

where

$$\mathcal{E}(t,0) = e^{-ct-c\int_0^t \|Y_N(r)\|_3^2 dr}.$$

From the definition of σ, we deduce

$$\mathbb{E}\int_0^\sigma \|\beta_N(t)\|_4^2 dt \le \|h\|_3^2 \exp\left(c(K_0+1) + c\Upsilon\right),$$

which yields Lemma 5.3. ■

5.2.3 Proof of (5.52)

Let $\psi \in C^\infty(\mathbb{R};[0,1])$ be such that

$$\psi = \begin{cases} 0, & \text{on } (\mathrm{K}_0+1,\infty), \\ 1, & \text{on } (-\infty,\mathrm{K}_0). \end{cases}$$

For process Y, set

$$\psi_Y = \psi\left(\int_0^\Upsilon \|Y(s)\|_3^2 ds\right).$$

Remark that

$$\left|\mathbb{E}\left(g(Y_N(\Upsilon,y_0^2)) - g(Y_N(\Upsilon,y_0^1))\right)\right| \le I_0 + |g|_\infty(I_1+I_2), \tag{5.60}$$

where

$$\begin{cases} I_0 = \left|\mathbb{E}\left(g(Y_N(\Upsilon,y_0^2))\psi_{Y_N(\cdot,y_0^2)} - g(Y_N(\Upsilon,y_0^1))\psi_{Y_N(\cdot,y_0^1)}\right)\right|, \\ I_i = \mathbb{P}\left(\int_0^\Upsilon \|Y_N(s,y_0^i)\|_3^2 ds > K_0\right). \end{cases}$$

For $\theta \in [1,2]$, set

$$\begin{cases} y_0^\theta = (2-\theta)y_0^1 + (\theta-1)y_0^2, \quad Y_\theta = Y_N(\cdot,y_0^\theta), \quad v_\theta = v_N(\cdot,y_0^\theta), \quad S_\theta = S_N(\cdot,y_0^\theta), \\ \beta_\theta(t) = \beta_N(t,0,y_0^\theta), \quad \eta_\theta(t) = \eta_N(t,0,y_0^\theta), \quad \sigma_\theta = \sigma(Y_\theta), \end{cases}$$

where σ is defined in (5.58). For better readability, the dependence on N has been omitted. Setting

$$h = y_0^2 - y_0^1,$$

we have

$$I_0 \le \int_1^2 |J_\theta| d\theta, \quad J_\theta = \left(\nabla_{y_0^\theta}\mathbb{E}(g(Y_\theta(\Upsilon))\psi_{Y_\theta}), h\right).$$

To estimate J_θ, applying a truncated Bismut-Elworthy-Li formula, similar to [26], we have

$$J_\theta = \frac{1}{\Upsilon} J'_{\theta,1} + 2J'_{\theta,2},$$

where

$$\begin{cases} J'_{\theta,1} = \mathbb{E}\left[g\left(Y_\theta(\Upsilon)\right) \psi_{Y_\theta} \int_0^{\sigma_\theta \wedge \Upsilon} \left(\Psi^{-1}(Y_\theta(t)) \cdot \beta_\theta(t) \cdot h, dW(t) \right) \right], \\ J'_{\theta,2} = \mathbb{E}\left[g(Y_\theta(\Upsilon)) \psi'_{Y_\theta} \int_0^{\sigma_\theta \wedge \Upsilon} (1 - \frac{t}{\Upsilon}) \left(A^{\frac{3}{2}} Y_\theta(t), A^{\frac{3}{2}} (\beta_\theta(t) \cdot h) dt \right) \right], \\ \psi'_{Y_\theta} = \psi' \left(\int_0^{\Upsilon} \|Y_\theta(s)\|_3^2 ds \right). \end{cases}$$

It follows from **Hypothesis H1** that

$$|J'_{\theta,1}| \le |g|_\infty \kappa \sqrt{\mathbb{E} \int_0^{\sigma_\theta \wedge \Upsilon} \|\beta_\theta(t) \cdot h\|_4^2 dt},$$

and from Hölder inequality that

$$|J'_{\theta,2}| \le |g|_\infty |\psi'|_\infty \sqrt{\mathbb{E} \int_0^{\sigma_\theta \wedge \Upsilon} \|Y_\theta(t)\|_3^2 dt} \sqrt{\mathbb{E} \int_0^{\sigma_\theta \wedge \Upsilon} \|\beta_\theta(t) \cdot h\|_3^2 dt}.$$

Hence for any $\Upsilon < 1$,

$$|J_\theta| \le c(\kappa, \mathbb{T}^3)|g|_\infty \frac{1}{\Upsilon} \sqrt{\mathbb{E} \int_0^{\sigma_\theta \wedge \Upsilon} \|\beta_\theta(t) \cdot h\|_4^2 dt}. \tag{5.61}$$

Combining (5.61) and Lemma 5.3, we have

$$|J_\theta| \le c(\kappa, \mathbb{T}^3)|g|_\infty \frac{\|h\|_3}{\Upsilon},$$

which yields,

$$I_0 \le c(\kappa, \mathbb{T}^3)|g|_\infty \sqrt{\Upsilon}.$$

Since $\kappa \Upsilon^3 \le \kappa \Upsilon$, we can apply Lemma 5.2 to control $I_1 + I_2$ in (5.60) if (5.51) holds. Hence (5.52) follows provided (5.51) holds, which yields Proposition 5.2. ■

5.3 Time of Return in a Small Ball of $\mathbb{H}_3$

Assume that **Hypothesis H1** holds. Let $N \in \mathbb{N}$ and $\Upsilon, \delta, Z_1, Z_2$ be as in Proposition 5.2. Let $(W, \tilde{W})$ be a couple of independent cylindrical Wiener processes on $H \times H$. Denote by $Y_N(\cdot, y_0)$ and $\tilde{Y}_N(\cdot, y_0)$ the solution of (4.33) associated to W and $\tilde{W}$, respectively. We construct a couple of random variables $(V_1, V_2) = (V_1(y_0^1, y_0^2), V_2(y_0^1, y_0^2))$ on $P_N H \times P_N H$ as follows

$$(V_1, V_2) = \begin{cases} (Y_N(\cdot, y_0), Y_N(\cdot, y_0)), & \text{if } y_0^1 = y_0^2 = y_0, \\ (Z_1(y_0^1, y_0^2), Z_2(y_0^1, y_0^2)), & \text{if } (y_0^1, y_0^2) \in B_{\mathbb{H}_3 \times \mathbb{H}_3}(0, \delta) \backslash \{y_0^1 = y_0^2\}, \\ (Y_N(\cdot, y_0^1), \tilde{Y}_N(\cdot, y_0^2)), & \text{else}. \end{cases} \tag{5.62}$$

Then, we can construct (Y^1, Y^2) by induction on $\Upsilon\mathbb{N}$. Indeed, firstly setting $Y^i(0) = y_0^i$ for $i = 1, 2$. Then, assuming that we have constructed (Y^1, Y^2) on $\{0, \Upsilon, 2\Upsilon, ..., n\Upsilon\}$, taking (V_1, V_2) as above independent of (Y^1, Y^2) and setting

$$Y^i((n+1)\Upsilon) = V_i(Y^1(n\Upsilon), Y^2(n\Upsilon)) \quad \text{for } i = 1, 2.$$

It follows that (Y^1, Y^2) is a discrete strong Markov process and a coupling of $(\mathcal{D}(Y_N(\cdot, y_0^1)), \mathcal{D}(Y_N(\cdot, y_0^2)))$ on $\Upsilon\mathbb{N}$. Moreover, if (Y^1, Y^2) are coupled at time $n\Upsilon$, then they remain coupled for any time after.

Setting

$$\tau = \inf\left\{t \in \Upsilon\mathbb{N}\backslash\{0\} \mid \|Y^1(t)\|_3^2 \vee \|Y^2(t)\|_3^2 \leq \delta\right\}. \tag{5.63}$$

The aim of this section is to establish the following Proposition 5.3.

Proposition 5.3. *Assume that **Hypothesis H1** holds. There exist* $\alpha = \alpha(\Psi, \mathbb{T}^3, \varepsilon_0, \delta) > 0$ *and* $K'' = K''(\Psi, \mathbb{T}^3, \varepsilon_0, \delta)$ *such that for any* $(y_0^1, y_0^2) \in H \times H$ *,*

$$\mathbb{E}(e^{\alpha\tau}) \leq K''(1 + |y_0^1|^2 + |y_0^2|^2).$$

The proof is postponed to Section 5.3.4, which is based on the following five lemmas.

Refer to [26], we know the following property of $Z(t)$ defined in Lemma 5.4.

Lemma 5.4. *Assume that **Hypothesis H1** holds. For any* $t, M > 0$*, there exists* $p_0(t, M) = p_0(t, M, \varepsilon_0, \{|\Psi_n|_\infty\}_n, \mathbb{T}^3) > 0$ *such that for any adapted process* Y*,*

$$\mathbb{P}\left(\sup_{(0,t)} \|Z(s)\|_3^2 \leq M\right) \geq p_0(t, M),$$

where

$$Z(t) = \int_0^t e^{-A(t-s)}\Psi(Y(s))dW(s).$$

Using this estimation, we can estimate the moment of the first time in a small ball in H. Let $\delta_3 > 0$. We set

$$\tau_{L^2} = \tau \wedge \inf\left\{t \in \Upsilon\mathbb{N} \mid |Y^1(t)|^2 \vee |Y^2(t)|^2 \leq \delta_3\right\}.$$

Lemma 5.5. *Assume that **Hypothesis H1** holds. Then, for any* $\delta_3 > 0$*, there exist* $C_3(\delta_3)$ *and* $\gamma_3(\delta_3)$ *such that for any* $(y_0^1, y_0^2) \in H \times H$*,*

$$\mathbb{E}(e^{\gamma_3\tau_{L^2}}) \leq C_3\left(1 + |y_0^1|^2 + |y_0^2|^2\right).$$

Proof of Lemma 5.5 Recall (5.38)

$$\mathbb{E}|Y_N(t)|^2 \leq e^{-\mu_1 t}|y_0|^2 + \frac{\kappa}{\mu_1}.$$

Since (Y^1, Y^2) is a coupling of $\left(\mathcal{D}(Y_N(\cdot, y_0^1)), \mathcal{D}(Y_N(\cdot, y_0^2))\right)$ on $\Upsilon\mathbb{N}$, we obtain

$$\mathbb{E}\left(|Y^1(n\Upsilon)|^2 + |Y^2(n\Upsilon)|^2\right) \leq e^{-\mu_1 n\Upsilon}(|y_0^1|^2 + |y_0^2|^2) + 2\frac{\kappa}{\mu_1}.$$

Since (Y^1, Y^2) is a strong Markov process, it can be deduced that there exist C_7 and γ_7 such that

$$\mathbb{E}\left(e^{\gamma_7 \tau'_{L^2}}\right) \leq C_7(1 + |y_0^1|^2 + |y_0^2|^2), \tag{5.64}$$

where

$$\tau'_{L^2} = \inf\left\{t \in \Upsilon\mathbb{N}\backslash\{0\} \mid |Y^1(t)|^2 + |Y^2(t)|^2 \leq 4\kappa\right\}.$$

Taking into account (5.64), a standard method shows that, in order to establish Lemma 5.5 it is sufficient to prove that there exist $(p_8(\delta_3, t), \Upsilon_8(\delta_3))$ such that

$$\mathbb{P}\left(|Y_N(t, y_0)|^2 \leq \delta_3\right) \geq p_8 > 0, \tag{5.65}$$

provided $N \in \mathbb{N}, t \geq \Upsilon_8(\delta_3)$, $|y_0|^2 \leq 4\kappa$ and $\Upsilon_8(\delta_3)$ is independent of y_0.

Setting

$$Z(t) = \int_0^t e^{-A(t-s)}\Psi(Y(s))dW(s), \quad X_N = Y_N - P_N Z,$$

$$N(\omega) = \sup_{(0,t)} \|Z(s,\omega)\|_3^2 \quad \text{for } \omega \in \Omega.$$

Assume that there exist $M_8(\delta_3) > 0$ and $\Upsilon_8(\delta_3)$ such that for $\omega \in \Omega$,

$$N(\omega) \leq M_8(\delta_3) \wedge \frac{\delta_3}{4} \text{ implies } \quad |X_N(t,\omega)|^2 \leq \frac{\delta_3}{4}, \tag{5.66}$$

provided $t \geq \Upsilon_8(\delta_3)$ and $|y_0|^2 \leq 4\kappa$. Then (5.65) results from Lemma 5.4 with $M = M_8(\delta_3) \wedge \frac{\delta_3}{4}$.

We now prove (5.66). From (5.36), we know

$$\begin{cases} \dfrac{\partial X_N}{\partial t} + P_N(v_N \cdot \nabla_H)(X_N + P_N Z) + P_N\Phi(v_N)\dfrac{\partial(X_N + P_N Z)}{\partial z} + P_N G(X_N + P_N Z) + AX_N = 0, \\ X_N(0) = P_N y_0. \end{cases}$$

Let

$$X_N = (\omega_N, g_N) = (v_N, S_N) - (P_N^1 Z_1, P_N^2 Z_2),$$

where

$$Z_1(t) = \int_0^t e^{-A_1(t-s)}\phi(v_N(s), S_N(s))dW_1(s), \quad Z_2(t) = \int_0^t e^{-A_2(t-s)}\varphi(v_N(s), S_N(s))dW_2(s).$$

For $\omega \in \Omega$, setting

$$N_1(\omega) = \sup_{(0,t)} \|Z_1(s,\omega)\|_3^2, \quad N_2(\omega) = \sup_{(0,t)} \|Z_2(s,\omega)\|_3^2,$$

we have

$$N_1(\omega) \vee N_2(\omega) \leq N(\omega).$$

From (5.53) and (5.54), we have

$$\begin{aligned} &\frac{\partial \omega_N}{\partial t} + \Big((\omega_N + Z_1) \cdot \nabla_H\Big)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z} \\ &\qquad + fk \times (\omega_N + Z_1) + \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^z \nabla_H(g_N + Z_2)dz' - \Delta\omega_N - \frac{\partial^2 \omega_N}{\partial z^2} = 0, \end{aligned} \tag{5.67}$$

$$\frac{\partial g_N}{\partial t} + [(\omega_N + Z_1)\cdot\nabla_H](g_N + Z_2) + \Phi(\omega_N + Z_1)\frac{\partial(g_N + Z_2)}{\partial z} - \Delta g_N - \frac{\partial^2 g_N}{\partial z^2} = 0. \tag{5.68}$$

Taking the scalar product of (5.67) with ω_N, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d|\omega_N|^2}{dt} + \|\omega_N\|^2 \quad &= \quad -\Big(\omega_N, ((\omega_N + Z_1)\cdot\nabla_H)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}\Big)\\ &\quad -\Big(\omega_N, fk\times(\omega_N + Z_1)\Big) - \Big(\omega_N, \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz'\Big),\end{aligned}$$

by integration by parts

$$\left(\omega_N, \Big((\omega_N + Z_1)\cdot\nabla_H\Big)\omega_N + \Phi(\omega_N + Z_1)\frac{\partial\omega_N}{\partial z}\right) = 0,$$

and

$$(\omega_N, fk\times\omega_N) = 0, \quad (\omega_N, \nabla_H p_b) = 0,$$

furthermore, by Lemma 2.4, Hölder inequality and the Young inequality, we have

$$\left|\Big(\omega_N, ((\omega_N + Z_1)\cdot\nabla_H)Z_1 + \Phi(\omega_N + Z_1)\frac{\partial Z_1}{\partial z}\Big)\right| \le c\|Z_1\|_3\|\omega_N\|^2 + \frac{1}{8}|\omega_N|^2 + c\|Z_1\|_3^4 + \frac{1}{8}\|\omega_N\|^2,$$

and

$$|(\omega_N, fk\times Z_1)| \le \frac{1}{8}|\omega_N|^2 + c|Z_1|^2,$$

$$\left|\Big(\omega_N, \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz'\Big)\right| \le \frac{1}{2}\|\omega_N\|^2 + \frac{1}{C_0}|g_N|^2 + \frac{1}{C_0}|Z_2|^2.$$

Combining all inequalities in the above, we obtain

$$\begin{aligned}\frac{d|\omega_N|^2}{dt} + \frac{1}{2}\|\omega_N\|^2 &\le c\|Z_1\|_3\|\omega_N\|^2 + c\|Z_1\|_3^4 + \frac{1}{C_0}|g_N|^2 + \frac{1}{C_0}|Z_2|^2 + c|Z_1|^2\\ &\le cM_8^{\frac{1}{2}}\|\omega_N\|^2 + cM_8 + \frac{2}{C_0}|g_N|^2.\end{aligned} \tag{5.69}$$

Similarly, taking the scalar product on both sides of (5.68) with g_N, using Hölder inequality and Sobolev embedding theorem, it follows that

$$\frac{d|g_N|^2}{dt} + \frac{3}{2}\|g_N\|^2 \le cM_8|\omega_N|^2 + cM_8^2|g_N|^2 + 2cM_8^2. \tag{5.70}$$

Since $C_0 \ge \frac{2}{\lambda_1}$, then $\frac{2}{C_0}|g_N|^2 \le \lambda_1|g_N|^2 \le \|g_N\|^2$. When M_8 is sufficiently small, combining (5.69) and (5.70), we obtain

$$\frac{d|X_N|^2}{dt} + \frac{1}{4}\|X_N\|^2 \le cM_8 \quad \text{on } (0,t). \tag{5.71}$$

Applying Gronwall inequality, we get

$$|X_N(t)|^2 \le e^{-\frac{\mu_1}{4}t}|y_0|^2 + \frac{cM_8}{\mu_1}.$$

Then, we deduce from $|y_0|^2 \leq 4\kappa$ that

$$|X_N(t)|^2 \leq 4\kappa e^{-\frac{\mu_1}{4}t} + \frac{cM_8}{\mu_1}.$$

Choosing t sufficiently large and M_8 sufficiently small, we obtain (5.66), which yields Lemma 5.5 holds. Indeed, when (5.66) holds,

$$\begin{aligned}
\mathbb{P}\left(|Y_N(t,y_0)|^2 \leq \delta_3\right) &\geq \mathbb{P}\left(|X_N(t,y_0)|^2 \leq \frac{\delta_3}{4}, \quad \sup_{(0,t)} \|Z(s)\|_3^2 \leq M_8(\delta_3) \wedge \frac{\delta_3}{4}\right) \\
&\geq \mathbb{P}\left(\sup_{(0,t)} \|Z(s)\|_3^2 \leq M_8(\delta_3) \wedge \frac{\delta_3}{4}\right) \\
&\geq p_0\left(t, M_8(\delta_3) \wedge \frac{\delta_3}{4}\right) > 0,
\end{aligned}$$

let $\tilde{M}_8(\delta_3) = M_8(\delta_3) \wedge \frac{\delta_3}{4}$ and choose $p_8(t,\delta_3) = p_0(t, \tilde{M}_8(\delta_3))$, which yields (5.65), then Lemma 5.5 holds. ■

In the following, we need to get a finer estimate in order to control the time necessary to enter a ball in stronger topology. Three steps are divided during the travel from H to $\mathbb{H}_3$.

5.3.1 From H to $\mathbb{H}_1$

Lemma 5.6. *Assume that* ***Hypothesis H1*** *holds. Then, for any $\delta_4 > 0$, there exist $p_4(\delta_4)$ and $R_4(\delta_4) > 0$ such that for any y_0 verifying $|y_0|^2 \leq R_4$, we have for any $\Upsilon \leq 1$,*

$$\mathbb{P}\left(\|Y_N(\Upsilon, y_0)\|^2 \leq \delta_4\right) \geq p_4.$$

Proof of Lemma 5.6 Using the decomposition $X_N = Y_N - P_N Z$ defined in Sect.3.1 and setting

$$N(\omega) = \sup_{(0,\Upsilon)} \|Z(s,\omega)\|_3^2 \quad \text{for } \omega \in \Omega.$$

Let $\delta_4 > 0$, assume that there exist $M_9(\delta_4) > 0$, $R_4(\delta_4) > 0$, such that for $\omega \in \Omega$,

$$N(\omega) \leq M_9(\delta_4) \wedge \frac{\delta_4}{4} \text{ implies } \quad \|X_N(\Upsilon, \omega, y_0)\|^2 \leq \frac{\delta_4}{4},$$

provided $|y_0|^2 \leq R_4(\delta_4)$. Then, Lemma 5.6 results from Lemma 5.4 with $M = M_9(\delta_4) \wedge \frac{\delta_4}{4}$.
Integrating (5.71),

$$\frac{1}{4\Upsilon} \int_0^{\Upsilon} \|X_N(t)\|^2 dt \leq \frac{1}{\Upsilon}|y_0|^2 + cM_8,$$

which yields, by Chebyshev inequality,

$$\lambda\left(t \in (0,\Upsilon) \mid \|X_N(t)\|^2 \leq \frac{8}{\Upsilon}|y_0|^2 + 8cM_8\right) \geq \frac{\Upsilon}{2}, \tag{5.72}$$

where λ denotes the Lebesgue measure on $(0, \Upsilon)$. Setting

$$\tau_{\mathbb{H}_1} = \inf\Big\{t \in (0, \Upsilon) \mid \|X_N(t)\|^2 \le \frac{8}{\Upsilon}|y_0|^2 + 8cM_8\Big\},$$

from (5.72) and the continuity of X_N, we deduce

$$\|X_N(\tau_{\mathbb{H}_1})\|^2 \le \frac{8}{\Upsilon}|y_0|^2 + 8cM_8. \tag{5.73}$$

Taking inner product with $\Delta\omega_N$ on both sides of (5.67) in $L^2(\mathbb{T}^3)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d|\nabla\omega_N|^2}{dt} + |\Delta\omega_N|_2^2 \quad &= \quad -\left(\Delta\omega_N, \left((\omega_N + Z_1)\cdot\nabla_H\right)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}\right) \\ &\quad -\left(\Delta\omega_N, fk\times(\omega_N + Z_1) + \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz'\right).\end{aligned}$$

Since

$$|(\Delta y, (x\cdot\nabla)z + (z\cdot\nabla)x)| \le c|\Delta y||\nabla z|^{\frac{1}{2}}|\Delta z|^{\frac{1}{2}}|\nabla x|,$$

we have

$$\begin{aligned}&\left|\left(\Delta\omega_N, ((\omega_N + Z_1)\cdot\nabla_H)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}\right)\right| \\ &\le c|\nabla\omega_N|^{\frac{3}{2}}|\Delta\omega_N|^{\frac{3}{2}} + c|\Delta Z_1|_2|\Delta\omega_N|^2 + c|\Delta Z_1|^2|\Delta\omega_N| + c|\nabla\omega_N||\Delta\omega_N|^2 + c|\Delta Z_1||\Delta\omega_N|^2.\end{aligned}$$

By Hölder inequality and the Young inequality, we have

$$|(\Delta\omega_N, fk\times(\omega_N + Z_1)| \le c|Z_1||\Delta\omega_N|,$$

$$\left|\left(\Delta\omega_N, \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz'\right)\right| \le \frac{1}{2}|\Delta\omega_N|^2 + \frac{1}{C_0}\|g_N\|^2 + \frac{1}{C_0}\|Z_2\|^2.$$

Thus, it follows

$$\frac{d\|\omega_N\|^2}{dt} + \|\omega_N\|_2^2 \le c\|\omega_N\|^6 + cM_9 + 4cM_9^{\frac{1}{2}}\|\omega_N\|_2^2 + \frac{2}{C_0}\|g_N\|^2 + c\|\omega_N\|^2\|\omega_N\|_2^2. \tag{5.74}$$

Similarly, taking inner product with Δg_N on both sides of (5.68) in $L^2(\mathbb{T}^3)$, we obtain

$$\begin{aligned}\frac{d\|g_N\|^2}{dt} + \frac{3}{2}\|g_N\|_2^2 \quad &\le \quad c\|g_N\|^2\|\omega_N\|^2\|\omega_N\|_2^2 + cM_9^2 + cM_9\|\omega_N\|_2^2 \\ &\quad + cM_9^{\frac{1}{2}}\|g_N\|_2^2 + c\|\omega_N\|^4\|g_N\|^2.\end{aligned} \tag{5.75}$$

Since $C_0 \ge \frac{2}{\lambda_1}$, then $\frac{2}{C_0}\|g_N\|^2 \le \lambda_1\|g_N\|^2 \le \|g_N\|_2^2$. When M_9 is sufficiently small, combining (5.74) and (5.75),

$$\frac{d\|X_N\|^2}{dt} + \|X_N\|_2^2 \le c\|\omega_N\|^6 + c\|\omega_N\|^2\|\omega_N\|_2^2 + c\|g_N\|^2\|\omega_N\|^2\|\omega_N\|_2^2 + cM_9.$$

Then

$$\frac{d\|X_N\|^2}{dt} + \frac{1}{4}\|X_N\|_2^2 + \left(\frac{1}{8} - c\|X_N\|^2 - c\|X_N\|^4\right)\|\omega_N\|_2^2 \le c\|X_N\|^2(\|X_N\|^4 - 4K_1^2) + cM_9,$$

where $K_1 = \sqrt{\frac{\mu_1}{8c}}$. Setting

$$\sigma_{\mathbb{H}_1} = \inf\left\{t \in (\tau_{\mathbb{H}_1}, \Upsilon) \mid \|X_N(t)\|^2 > 2K_1 \wedge \frac{1}{\sqrt{8c}}\right\},$$

denote by $2\tilde{K}_1 = 2K_1 \wedge \frac{1}{\sqrt{8c}}$. Remark that on $(\tau_{\mathbb{H}_1}, \sigma_{\mathbb{H}_1})$, we have

$$\frac{d\|X_N\|^2}{dt} + \frac{1}{4}\|X_N\|_2^2 \leq cM_9. \tag{5.76}$$

Integrating (5.76), we obtain

$$\begin{aligned} \|X_N(\sigma_{\mathbb{H}_1})\|^2 + \frac{1}{4}\int_{\tau_{\mathbb{H}_1}}^{\sigma_{\mathbb{H}_1}} \|X_N(t)\|_2^2 dt &\leq \|X_N(\tau_{\mathbb{H}_1})\|^2 + cM_9(\sigma_{\mathbb{H}_1} - \tau_{\mathbb{H}_1}) \quad (5.77)\\ &\leq \|X_N(\tau_{\mathbb{H}_1})\|^2 + cM_9\Upsilon \\ &\leq \|X_N(\tau_{\mathbb{H}_1})\|^2 + cM_9. \end{aligned}$$

From (5.73) and (5.77), we obtain that, for M_8, M_9 and $|y_0|^2$ sufficiently small,

$$\|X_N(\sigma_{\mathbb{H}_1})\|^2 \leq \frac{\delta_4}{4} \wedge \tilde{K}_1,$$

which yields $\sigma_{\mathbb{H}_1} = \Upsilon$. It follows that

$$\|X_N(\Upsilon)\|^2 \leq \frac{\delta_4}{4},$$

provided M_8, M_9 and $|y_0|^2$ sufficiently small. Since

$$\begin{aligned} \mathbb{P}\left(\|Y_N(\Upsilon, y_0)\|^2 \leq \delta_4\right) &\geq \mathbb{P}\left(\|X_N(\Upsilon, y_0)\|^2 \leq \frac{\delta_4}{4}, \quad \sup_{(0,\Upsilon)} \|Z(s)\|_3^2 \leq M_9(\delta_4) \wedge \frac{\delta_4}{4}\right) \\ &\geq \mathbb{P}\left(\sup_{(0,\Upsilon)} \|Z(s)\|_3^2 \leq M_9(\delta_4) \wedge \frac{\delta_4}{4}\right) \\ &\geq p_0\left(\Upsilon, M_9(\delta_4) \wedge \frac{\delta_4}{4}\right) > 0, \end{aligned}$$

let $\tilde{M}_9(\delta_4) = M_9(\delta_4) \wedge \frac{\delta_4}{4}$ and choose $p_4(\delta_4) = p_0(\Upsilon, \tilde{M}_9(\delta_4))$, we obtain Lemma 5.6. ■

5.3.2 From $\mathbb{H}_1$ to $\mathbb{H}_2$

Lemma 5.7. *Assume that **Hypothesis H1** holds. Then, for any $\delta_5 > 0$, there exist $p_5(\delta_5)$ and $R_5(\delta_5) > 0$ such that for any y_0 verifying $\|y_0\|^2 \leq R_5$, we have for any $\Upsilon \leq 1$,*

$$\mathbb{P}\left(\|Y_N(\Upsilon, y_0)\|_2^2 \leq \delta_5\right) \geq p_5.$$

Proof of Lemma 5.7 When M_9 sufficiently small and $\|y_0\|^2 + cM_9 \leq \tilde{K}_1$, we have $\tau_{\mathbb{H}_1} = 0$ and $\sigma_{\mathbb{H}_1} = \Upsilon$, then it follows from (5.76) that

$$\frac{1}{4}\int_0^{\Upsilon} \|X_N(t)\|_2^2 dt \leq \|y_0\|^2 + cM_9.$$

Applying the same argument as in subsection 6.3.3, it is easy to deduce that there exists a stopping time $\tau_{\mathbb{H}_2} \in (0, \Upsilon)$ such that

$$\|X_N(\tau_{\mathbb{H}_2})\|_2^2 \le \frac{8}{\Upsilon}(\|y_0\|^2 + cM_9), \tag{5.78}$$

provided M_9 and $\|y_0\|^2$ are sufficiently small.

Similar to the above, let $\delta_5 > 0$, assume that there exist $M_{10}(\delta_5) > 0$ and $R_5(\delta_5) > 0$ such that for $\omega \in \Omega$,

$$N(\omega) \le M_{10}(\delta_5) \wedge \frac{\delta_5}{4} \text{ implies } \quad \|X_N(\Upsilon, \omega, y_0)\|_2^2 \le \frac{\delta_5}{4},$$

provided $\|y_0\|^2 \le R_5(\delta_5)$. Then, Lemma 5.7 results from Lemma 5.4 with $M = M_{10}(\delta_5) \wedge \frac{\delta_5}{4}$.

Applying the operator ∇ to (5.67), multiplying the resulting equation by $-\nabla\Delta\omega_N$, then integrating over $\mathbb{T}^3$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d|\Delta\omega_N|^2}{dt} + |\nabla\Delta\omega_N|^2 \quad &= \quad \left(\nabla\Delta\omega_N, \nabla[\Big((\omega_N + Z_1)\cdot\nabla_H\Big)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}]\right) \\ &\quad \left(\nabla\Delta\omega_N, \nabla[fk \times (\omega_N + Z_1) + \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz']\right).\end{aligned}$$

Since

$$|(\nabla\Delta y, \nabla[(x\cdot\nabla_H)z + (z\cdot\nabla_H)x])| \le c|\nabla\Delta y||\Delta z||\Delta x| \le \varepsilon|\nabla\Delta y|^2 + c(|\Delta z|^4 + |\Delta x|^4),$$

we obtain

$$\begin{aligned}&\left|\left(\nabla\Delta\omega_N, \nabla[\Big((\omega_N + Z_1)\cdot\nabla_H\Big)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}]\right)\right| \\ &\le c|\nabla\Delta\omega_N|^{\frac{3}{2}}|\Delta\omega_N|^{\frac{1}{2}}|\nabla\omega_N| + c|\nabla\Delta\omega_N||\Delta\omega_N|^{\frac{3}{2}}|\nabla\omega_N|^{\frac{1}{2}} + c|\nabla\Delta\omega_N||\Delta\omega_N|^{\frac{1}{2}}|\nabla\omega_N|^{\frac{1}{2}}|\Delta Z_1| \\ &\quad + c|\nabla\Delta\omega_N|^2|\nabla\Delta Z_1| + c|\nabla\Delta\omega_N||\nabla\Delta Z_1|^2 + c|\nabla\Delta\omega_N|^{\frac{3}{2}}|\Delta\omega_N|^{\frac{3}{2}} + c|\Delta Z_1||\Delta\omega_N|^2 \\ &\quad + c|\Delta Z_1|^2|\Delta\omega_N| + c|\nabla\omega_N||\Delta\omega_N|^2 + c|\Delta Z_1||\Delta\omega_N|^2 \\ &\le \varepsilon|\nabla\Delta\omega_N|^2 + c|\Delta\omega_N|^6 + c|\nabla\Delta Z_1|^2|\Delta\omega_N|^2 + c|\nabla\Delta Z_1|^4.\end{aligned}$$

By Hölder inequality and the Young inequality, we have

$$|(\nabla\Delta\omega_N, \nabla Z_1)| \le c|\nabla Z_1||\nabla\Delta\omega_N|,$$

$$\left|\left(\nabla\Delta\omega_N, \nabla[\nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz']\right)\right| \le \frac{1}{2}|\nabla\Delta\omega_N|^2 + \frac{1}{C_0}|\Delta g_N|^2 + \frac{1}{C_0}|\Delta Z_2|^2.$$

Combining the above inequalities, applying Hölder inequality and the Young inequality, we obtain

$$\frac{d\|\omega_N\|_2^2}{dt} + \|\omega_N\|_3^2 \le c\|\omega_N\|_2^6 + c\|\omega_N\|_2^4 + cM_{10}^{\frac{1}{2}}\|\omega_N\|_3^2 + \frac{2}{C_0}\|g_N\|_2^2 + cM_{10}^2 + \varepsilon\|\omega_N\|_3^2. \tag{5.79}$$

Similarly, applying the operator ∇ to (5.68), multiplying the resulting equation by $-\nabla\Delta g_N$, then integrating over $\mathbb{T}^3$, it follows that

$$\begin{aligned}\frac{d\|g_N\|_2^2}{dt} + 2\|g_N\|_3^2 \quad &\le \quad \varepsilon\|g_N\|_3^2 + c\|g_N\|_2^2\|\omega_N\|_2^4 + c\|g_N\|_2^2\|\omega_N\|_2^2 \\ &\quad + cM_{10}\|\omega_N\|_3^2 + cM_{10}\|g_N\|_3^2 + cM_{10}^2.\end{aligned} \tag{5.80}$$

Choosing M_{10} small enough, since $\frac{2}{C_0}\|g_N\|_2^2 \le \lambda_1\|g_N\|_2^2 \le \|g_N\|_3^2$, by (5.79) and (5.80), we obtain

$$\frac{d\|X_N\|_2^2}{dt} + \frac{1}{4}\|X_N\|_3^2 \le c\|X_N\|_2^2\left(\|X_N\|_2^2 + \|X_N\|_2^4 - 4K_2^2\right) + cM_{10}. \tag{5.81}$$

where K_2 is defined similar to the above. Setting $\sigma_{\mathbb{H}_2} = \inf\{t \in (\tau_{\mathbb{H}_2}, \Upsilon) \mid \|X_N\|_2^2 \ge 2K_2\}$, integrating (5.81) on $(\tau_{\mathbb{H}_2}, \sigma_{\mathbb{H}_2})$, we obtain

$$\|X_N(\sigma_{\mathbb{H}_2})\|_2^2 + \frac{1}{4}\int_{\tau_{\mathbb{H}_2}}^{\sigma_{\mathbb{H}_2}} \|X_N(t)\|_3^2 dt \le \|X_N(\tau_{\mathbb{H}_2})\|_2^2 + cM_{10}. \tag{5.82}$$

Combining (5.78) and (5.82). We obtain that, for M_9, M_{10} and $\|y_0\|^2$ sufficiently small,

$$\|X_N(\sigma_{\mathbb{H}_2})\|_2^2 \le \frac{\delta_5}{4} \wedge K_2.$$

It follows that $\sigma_{\mathbb{H}_2} = \Upsilon$ and

$$\|X_N(\Upsilon)\|_2^2 \le \frac{\delta_5}{4},$$

provided M_9, M_{10} and $\|y_0\|^2$ sufficiently small. Since

$$\begin{aligned}
\mathbb{P}\left(\|Y_N(\Upsilon, y_0)\|_2^2 \le \delta_5\right) &\ge \mathbb{P}\left(\|X_N(\Upsilon, y_0)\|_2^2 \le \frac{\delta_5}{4}, \quad \sup_{(0,\Upsilon)} \|Z(s)\|_3^2 \le M_{10}(\delta_5) \wedge \frac{\delta_5}{4}\right) \\
&\ge \mathbb{P}\left(\sup_{(0,\Upsilon)} \|Z(s)\|_3^2 \le M_{10}(\delta_5) \wedge \frac{\delta_5}{4}\right) \\
&\ge p_0\left(\Upsilon, M_{10}(\delta_5) \wedge \frac{\delta_5}{4}\right) > 0,
\end{aligned}$$

let $\tilde{M}_{10}(\delta_5) = M_{10}(\delta_5) \wedge \frac{\delta_5}{4}$ and choose $p_5(\delta_5) = p_0(\Upsilon, \tilde{M}_{10}(\delta_5))$, we obtain Lemma 5.7. ■

5.3.3 From $\mathbb{H}_2$ to $\mathbb{H}_3$

Lemma 5.8. *Assume that **Hypothesis H1** holds. Then, for any $\delta_6 > 0$, there exist $p_6(\delta_6)$ and $R_6(\delta_6) > 0$ such that for any y_0 verifying $\|y_0\|_2^2 \le R_6$, we have for any $\Upsilon \le 1$,*

$$\mathbb{P}\left(\|Y_N(\Upsilon, y_0)\|_3^2 \le \delta_6\right) \ge p_6.$$

Proof of Lemma 5.8 When M_{10} sufficiently small and $\|y_0\|_2^2 + cM_{10} \le K_2$, we have $\tau_{\mathbb{H}_2} = 0$ and $\sigma_{\mathbb{H}_2} = \Upsilon$. Taking into account (5.81), we obtain

$$\frac{1}{4}\int_0^{\Upsilon} \|X_N(t)\|_3^2 dt \le \|y_0\|_2^2 + cM_{10}.$$

Applying the same argument from Lemma 3.4, it's easy to know there exists a stopping time $\tau_{\mathbb{H}_3} \in (0, \Upsilon)$ such that

$$\|X_N(\tau_{\mathbb{H}_3})\|_3^2 \le \frac{8}{\Upsilon}(\|y_0\|_2^2 + cM_{10}). \tag{5.83}$$

Similar to the above, let $\delta_6 > 0$, assume that there exist $M_{11}(\delta_6) > 0$ and $R_6(\delta_6) > 0$ such that for $\omega \in \Omega$,

$$N(\omega) \le M_{11}(\delta_6) \wedge \frac{\delta_6}{4} \text{ implies } \quad \|X_N(\Upsilon, \omega, y_0)\|_3^2 \le \frac{\delta_6}{4},$$

provided $\|y_0\|_2^2 \le R_6(\delta_6)$. Then, Lemma 5.8 results from Lemma 5.4 with $M = M_{11}(\delta_6) \wedge \frac{\delta_6}{4}$.

Applying the operator Δ to (5.67), multiplying the resulting equation by $-\Delta^2 \omega_N$, then integrating over $\mathbb{T}^3$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d|\nabla\Delta\omega_N|^2}{dt} + |\Delta^2\omega_N|^2 \quad &= \quad \left(\Delta^2\omega_N, \Delta[\Big((\omega_N + Z_1)\cdot\nabla_H\Big)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}]\right) \\ &\quad + \left(\Delta^2\omega_N, \Delta[fk\times(\omega_N + Z_1) + \nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz']\right),\end{aligned}$$

Since

$$|(\Delta^2 y, \Delta[(x\cdot\nabla)z + (z\cdot\nabla)x])| \le c|\Delta^2 y||\nabla\Delta z||\nabla\Delta x| \le \varepsilon|\Delta^2 y|^2 + c(|\nabla\Delta z|^4 + |\nabla\Delta x|^4),$$

we obtain

$$\begin{aligned}&\left|\left(\Delta^2\omega_N, \Delta[((\omega_N + Z_1)\cdot\nabla_H)(\omega_N + Z_1) + \Phi(\omega_N + Z_1)\frac{\partial(\omega_N + Z_1)}{\partial z}]\right)\right| \\ &\le c|\Delta^2\omega_N|\|\omega_N\|_3\|\omega_N\|_2 + c|\Delta^2\omega_N||\nabla\Delta\omega_N||\nabla\Delta Z_1| + c|\Delta^2\omega_N||\nabla\Delta Z_1|^2 \\ &\quad + c|\Delta^2\omega_N||\nabla\Delta Z_1|^2 + c|\Delta^2\omega_N||\nabla\Delta\omega_N|^2 + c|\Delta^2\omega_N|^{\frac{3}{2}}|\nabla\Delta\omega_N|^{\frac{1}{2}}\|Z_1\|_2 \\ &\le \varepsilon|\Delta^2\omega_N|^2 + c|\nabla\Delta\omega_N|^4 + c|\nabla\Delta Z_1|^2 + c|\nabla\Delta Z_1|^4 + c|\nabla\Delta Z_1|^4|\nabla\Delta\omega_N|^2.\end{aligned}$$

By Hölder inequality and the Young inequality, we have

$$|(\Delta^2\omega_N, \Delta Z_1)| \le c|\Delta Z_1||\Delta^2\omega_N|,$$

$$\left|\left(\Delta^2\omega_N, \Delta[\nabla_H p_b - \frac{1}{\sqrt{C_0}}\int_{-1}^{z}\nabla_H(g_N + Z_2)dz']\right)\right| \le \frac{1}{2}|\Delta^2\omega_N|^2 + \frac{1}{C_0}|\nabla\Delta g_N|^2 + \frac{1}{C_0}|\nabla\Delta Z_2|^2.$$

Combining the above inequalities, we obtain

$$\frac{d\|\omega_N\|_3^2}{dt} + \|\omega_N\|_4^2 \le \varepsilon\|\omega_N\|_4^2 + c\|\omega_N\|_3^4 + cM_{11} + cM_{11}\|\omega_N\|_3^2 + \frac{2}{C_0}\|g_N\|_3^2. \tag{5.84}$$

Similarly, applying the operator Δ to (5.68), multiplying the resulting equation by $-\Delta^2 g_N$, then integrating over $\mathbb{T}^3$, it follows that

$$\frac{d\|g_N\|_3^2}{dt} + 2\|g_N\|_4^2 \le \varepsilon\|g_N\|_4^2 + c\|\omega_N\|_3^2\|g_N\|_3^2 + cM_{11}\|g_N\|_4^2 + cM_{11}\|\omega_N\|_4^2 + cM_{11}^2. \tag{5.85}$$

Choosing M_{11} sufficiently small, since $\frac{2}{C_0}\|g_N\|_3^2 \le \lambda_1\|g_N\|_3^2 \le \|g_N\|_4^2$, by (5.84) and (5.85), we obtain

$$\frac{d\|X_N\|_3^2}{dt} + \frac{3}{4}\|X_N\|_4^2 \le c\|X_N\|_3^4 + cM_{11},$$

then

$$\frac{d\|X_N\|_3^2}{dt} + \frac{1}{4}\|X_N\|_4^2 \le c\|X_N\|_3^2\left(\|X_N\|_3^2 - 2K_3\right) + cM_{11}, \tag{5.86}$$

where K_3 is defined similarly to the above. Setting $\sigma_{\mathbb{H}_3} = \inf\{t \in (\tau_{\mathbb{H}_3}, \Upsilon) \mid \|X_N(t)\|_3^2 \geq 2K_3\}$. Integrating (5.86) on $(\tau_{\mathbb{H}_3}, \sigma_{\mathbb{H}_3})$, we obtain

$$\|X_N(\sigma_{\mathbb{H}_3})\|_3^2 + \frac{1}{4}\int_{\tau_{\mathbb{H}_3}}^{\sigma_{\mathbb{H}_3}} \|X_N(t)\|_4^2 dt \leq \|X_N(\tau_{\mathbb{H}_3})\|_3^2 + cM_{11}.$$

Taking into account (5.83) and choosing M_{10}, M_{11} and $\|y_0\|_2^2$ sufficiently small, we obtain

$$\|X_N(\sigma_{\mathbb{H}_3})\|_3^2 \leq \frac{\delta_6}{4} \wedge K_3.$$

It follows that $\sigma_{\mathbb{H}_3} = \Upsilon$ and that

$$\|X_N(\Upsilon)\|_3^2 \leq \frac{\delta_6}{4},$$

provided M_{10}, M_{11} and $\|y_0\|_2^2$ sufficiently small. Since

$$\begin{aligned}
\mathbb{P}\left(\|Y_N(\Upsilon, y_0)\|_3^2 \leq \delta_6\right) &\geq \mathbb{P}\left(\|X_N(\Upsilon, y_0)\|_3^2 \leq \frac{\delta_6}{4}, \quad \sup_{(0,\Upsilon)} \|Z(s)\|_3^2 \leq M_{11}(\delta_6) \wedge \frac{\delta_6}{4}\right) \\
&\geq \mathbb{P}\left(\sup_{(0,\Upsilon)} \|Z(s)\|_3^2 \leq M_{11}(\delta_6) \wedge \frac{\delta_6}{4}\right) \\
&\geq p_0\left(\Upsilon, M_{11}(\delta_6) \wedge \frac{\delta_6}{4}\right) > 0,
\end{aligned}$$

let $\tilde{M}_{11}(\delta_6) = M_{11}(\delta_6) \wedge \frac{\delta_6}{4}$ and choose $p_6(\delta_6) = p_0(\Upsilon, \tilde{M}_{11}(\delta_6))$, we obtain Lemma 5.8. ■

5.3.4 Proof of Proposition 5.3

Setting

$$\begin{aligned}
&\delta_6 = \delta,\ \delta_5 = R_6(\delta_6), \delta_4 = R_5(\delta_5), \delta_3 = R_4(\delta_4), \\
&p_4 = p_4(\delta_4), p_5 = p_5(\delta_5), p_6 = p_6(\delta_6), p_1 = (p_4 p_5 p_6)^2.
\end{aligned}$$

By the definition of τ_{L^2},

$$|Y^1(\tau_{L^2})|^2 \vee |Y^2(\tau_{L^2})|^2 \leq R_4(\delta_4).$$

We prove this proposition by three cases.

The first case: $\|Y^1(\tau_{L^2})\|_3^2 \vee \|Y^2(\tau_{L^2})\|_3^2 \leq \delta$, which obviously yields

$$\mathbb{P}\left(\min_{k=0,1,2,3} \max_{i=1,2} \|Y^i(\tau_{L^2} + k\Upsilon)\|_3^2 \leq \delta \mid \left(Y^1(\tau_{L^2}), Y^2(\tau_{L^2})\right)\right) \geq p_1. \tag{5.87}$$

The second case: $Y^1(\tau_{L^2}) = Y^2(\tau_{L^2}) = y_0$ with $\|y_0\|_3^2 > \delta$. Combining Lemma 5.6, Lemma 5.7 and Lemma 5.8, we deduce from the strong Markov property of Y_N that

$$\mathbb{P}(\|Y_N(3\Upsilon, y_0)\|_3^2 \leq \delta) \geq p_6 p_5 p_4,$$

provided $|y_0|^2 \leq R_4$. In that case, $Y^1(\tau_{L^2} + 3\Upsilon) = Y^2(\tau_{L^2} + 3\Upsilon)$. Hence, since the law of $Y^1(\tau_{L^2} + 3\Upsilon)$ conditioned by $(Y^1(\tau_{L^2}), Y^2(\tau_{L^2})) = (y_0^1, y_0^2)$ is $\mathcal{D}(Y_N(3\Upsilon, y_0))$, it follows

$$\mathbb{P}\left(\max_{i=1,2} \|Y^i(\tau_{L^2} + 3\Upsilon)\|_3^2 \leq \delta \mid \left(Y^1(\tau_{L^2}), Y^2(\tau_{L^2})\right)\right) \geq p_6 p_5 p_4 \geq p_1,$$

then (5.87) holds.

The third case: $Y^1(\tau_{L^2}) \neq Y^2(\tau_{L^2})$ and $\|Y^1(\tau_{L^2})\|_3^2 \vee \|Y^2(\tau_{L^2})\|_3^2 > \delta$. In that case, $(Y^1(\tau_{L^2} + \Upsilon), Y^2(\tau_{L^2} + \Upsilon))$ conditioned by $(Y^1(\tau_{L^2}), Y^2(\tau_{L^2}))$ are independent. Since the law of $Y^i(\tau_{L^2} + \Upsilon)$ conditioned by $(Y^1(\tau_{L^2}), Y^2(\tau_{L^2})) = (y_0^1, y_0^2)$ is $\mathcal{D}(Y_N(\Upsilon, y_0^i))$, it follows from Lemma 5.6 that

$$\mathbb{P}\left(\max_{i=1,2} \|Y^i(\tau_{L^2} + \Upsilon)\|^2 \leq \delta_4 \mid \left(Y^1(\tau_{L^2}), Y^2(\tau_{L^2})\right)\right) \geq p_4^2. \tag{5.88}$$

Then, we distinguish three cases for $(Y^i(\tau_{L^2} + \Upsilon))_{i=1,2}$ similar to $(Y^i(\tau_{L^2}))_{i=1,2}$: $\|Y^1(\tau_{L^2} + \Upsilon)\|_3^2 \vee \|Y^2(\tau_{L^2} + \Upsilon)\|_3^2 \leq \delta$, $Y^1(\tau_{L^2} + \Upsilon) = Y^2(\tau_{L^2} + \Upsilon) = y_1$ with $\|y_1\|_3^2 > \delta$, $Y^1(\tau_{L^2} + \Upsilon) \neq Y^2(\tau_{L^2} + \Upsilon)$ and $\|Y^1(\tau_{L^2} + \Upsilon)\|_3^2 \vee \|Y^2(\tau_{L^2} + \Upsilon)\|_3^2 > \delta$. For the front two cases, using the method similar to the first and second case, respectively, and combining (5.88), we have (5.87) holds. For the last case, we know that $(Y^1(\tau_{L^2} + 2\Upsilon), Y^2(\tau_{L^2} + 2\Upsilon))$ conditioned by $(Y^1(\tau_{L^2} + \Upsilon), Y^2(\tau_{L^2} + \Upsilon))$ are independent. By Lemma 5.7, we have

$$\mathbb{P}\left(\min_{k=1,2} \max_{i=1,2} \|Y^i(\tau_{L^2} + k\Upsilon)\|_2^2 \leq \delta_5 \mid \left(Y^1(\tau_{L^2} + \Upsilon), Y^2(\tau_{L^2} + \Upsilon)\right)\right) \geq p_5^2, \tag{5.89}$$

provided

$$\max_{i=1,2} \|Y^i(\tau_{L^2} + \Upsilon)\|^2 \leq \delta_4.$$

Now, we distinguish three cases for $(Y^i(\tau_{L^2} + 2\Upsilon))_{i=1,2}$ similar to the above: $\|Y^1(\tau_{L^2} + 2\Upsilon)\|_3^2 \vee \|Y^2(\tau_{L^2} + 2\Upsilon)\|_3^2 \leq \delta$, $Y^1(\tau_{L^2} + 2\Upsilon) = Y^2(\tau_{L^2} + 2\Upsilon) = y_2$ with $\|y_2\|_3^2 > \delta$, $Y^1(\tau_{L^2} + 2\Upsilon) \neq Y^2(\tau_{L^2} + 2\Upsilon)$ and $\|Y^1(\tau_{L^2} + 2\Upsilon)\|_3^2 \vee \|Y^2(\tau_{L^2} + 2\Upsilon)\|_3^2 > \delta$. For the front two cases, we also can obtain (5.87). For the last case, $(Y^1(\tau_{L^2} + 3\Upsilon), Y^2(\tau_{L^2} + 3\Upsilon))$ conditioned by $(Y^1(\tau_{L^2} + 2\Upsilon), Y^2(\tau_{L^2} + 2\Upsilon))$ are independent. By Lemma 5.8,

$$\mathbb{P}\left(\min_{k=2,3} \max_{i=1,2} \|Y^i(\tau_{L^2} + k\Upsilon)\|_3^2 \leq \delta \mid \left(Y^1(\tau_{L^2} + 2\Upsilon), Y^2(\tau_{L^2} + 2\Upsilon)\right)\right) \geq p_6^2, \tag{5.90}$$

provided

$$\max_{i=1,2} \|Y^i(\tau_{L^2} + 2\Upsilon)\|_2^2 \leq \delta_5.$$

Combining (5.88), (5.89) and (5.90), we deduce (5.87) holds for the last case.

Thus, we have proved (5.87) is true almost surely. Integrating (5.87), we obtain

$$\mathbb{P}\left(\min_{k=0,1,2,3} \max_{i=1,2} \|Y^i(\tau_{L^2} + k\Upsilon)\|_3^2 \leq \delta\right) \geq p_1. \tag{5.91}$$

Combining Lemma 5.5 and (5.91), we conclude that Proposition 5.3 holds. ■

5.4 Proof of Theorem 3.2

As above explained, we only need to prove Proposition 5.1. Let $(y_0^1, y_0^2) \in \mathbb{H}_3 \times \mathbb{H}_3$. Let's recall the process (Y^1, Y^2) is defined at the beginning of Sect. 3. Let $\delta > 0$, $\Upsilon \in (0, 1)$ be as in Proposition 5.2 and τ defined in (5.63), setting

$$\tau_1 = \tau, \quad \tau_{k+1} = \inf\left\{t > \tau_k \mid \|Y^1(t)\|_3^2 \vee \|Y^2(t)\|_3^2 \leq \delta\right\}.$$

It can be deduced from the strong Markov property of (Y^1, Y^2) and Proposition 5.3 that

$$\mathbb{E}(e^{\alpha\tau_{k+1}}) \leq K''\mathbb{E}\left(e^{\alpha\tau_k}(1 + |Y^1(\tau_k)|^2 + |Y^2(\tau_k)|^2)\right),$$

which yields,

$$\begin{cases} \mathbb{E}(e^{\alpha\tau_{k+1}}) \leq cK''(1+2\delta)\mathbb{E}(e^{\alpha\tau_k}), \\ \mathbb{E}(e^{\alpha\tau_1}) \leq K''(1 + |y_0^1|^2 + |y_0^2|^2). \end{cases}$$

It follows that there exists $K > 0$ such that

$$\mathbb{E}(e^{\alpha\tau_k}) \leq K^k(1 + |y_0^1|^2 + |y_0^2|^2).$$

Hence, applying the Jensen inequality, we obtain, for any $\theta \in (0, 1)$,

$$\mathbb{E}(e^{\theta\alpha\tau_k}) \leq K^{\theta k}(1 + |y_0^1|^2 + |y_0^2|^2). \tag{5.92}$$

We deduce from Proposition 5.2 and from (5.62) that

$$\mathbb{P}\left(Y^1(\Upsilon) \neq Y^2(\Upsilon)\right) \leq \frac{1}{4},$$

provided (y_0^1, y_0^2) is in the ball of $\mathbb{H}_3 \times \mathbb{H}_3$ with radius δ. Setting

$$k_0 = \inf\left\{k \in \mathbb{N} \mid Y^1(\tau_k + \Upsilon) = Y^2(\tau_k + \Upsilon)\right\}.$$

By strong Markov property of (Y^1, Y^2), we have

$$\mathbb{P}(k_0 > n) \leq \left(\frac{1}{4}\right)^n, \tag{5.93}$$

which implies $k_0 < \infty$ almost surely. Let $\theta \in (0, 1)$, we deduce from Cauchy-Schwarz inequality that

$$\mathbb{E}(e^{\frac{\theta}{2}\alpha\tau_{k_0}}) = \sum_{n=1}^{\infty} \mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_n} I_{k_0=n}\right) \leq \sum_{n=1}^{\infty} \sqrt{\mathbb{P}(k_0 \geq n)\mathbb{E}(e^{\theta\alpha\tau_n})}.$$

Combining (5.92) and (5.93), we deduce

$$\mathbb{E}(e^{\frac{\theta}{2}\alpha\tau_{k_0}}) \leq \left(\sum_{n=0}^{\infty}\left(\frac{K^\theta}{2}\right)^n\right)(1 + |y_0^1|^2 + |y_0^2|^2).$$

Hence, choosing $\theta \in (0, 1)$ sufficiently small, we obtain that there exists $\gamma > 0$ independent of $N \in \mathbb{N}$ such that

$$\mathbb{E}(e^{\gamma\tau_{k_0}}) \leq 4(1 + |y_0^1|^2 + |y_0^2|^2). \tag{5.94}$$

Recall that if (Y^1, Y^2) are coupled at time $t \in \Upsilon\mathbb{N}$, then they remain coupled for any time after. Hence $Y^1(t) = Y^2(t)$ for $t > \tau_{k_0}$. It follows

$$\mathbb{P}(Y^1(n\Upsilon) \neq Y^2(n\Upsilon)) \leq 4e^{-\gamma n\Upsilon}(1 + |y_0^1|^2 + |y_0^2|^2).$$

Since $\Big(Y^1(n\Upsilon), Y^2(n\Upsilon)\Big)$ is a coupling of $\Big((\mathcal{P}^N_{n\Upsilon})^*\delta_{y_0^1}, (\mathcal{P}^N_{n\Upsilon})^*\delta_{y_0^2}\Big)$, we deduce from Lemma 5.1

$$\|(\mathcal{P}^N_{n\Upsilon})^*\delta_{y_0^1} - (\mathcal{P}^N_{n\Upsilon})^*\delta_{y_0^2}\|_{var} \le 4e^{-\gamma n\Upsilon}(1 + |y_0^1|^2 + |y_0^2|^2), \tag{5.95}$$

for any $n \in \mathbb{N}$ and any $(y_0^1, y_0^2) \in \mathbb{H}_3 \times \mathbb{H}_3$. Recall that the existence of an invariant measure $\mu_N \in \mathcal{P}(P_N H)$ can be justified by (5.39). Let $\lambda \in \mathcal{P}(H)$ and $t \in \mathbb{R}^+$. We set $n = \lfloor \frac{t}{\Upsilon} \rfloor$ and $C = 4e^{\gamma\Upsilon}$. Integrating (y_0^1, y_0^2) over $((\mathcal{P}^N_{t-n\Upsilon})^*\lambda) \otimes \mu_N$ in (5.95), we obtain

$$\|(\mathcal{P}^N_t)^*\lambda - \mu_N\|_{var} \le Ce^{-\gamma t}\left(1 + \int_H |y|^2 \lambda(dy)\right), \tag{5.96}$$

which establishes (5.47), then Proposition 5.1 holds.

We now explain why Proposition 5.1 implies Theorem 3.2. First of all, (3.32) is equivalent to

$$\left|\mathbb{E}_\lambda\Big(g(Y(t))\Big) - \int_H g(y)\mu(dy)\right| \le Ce^{-\gamma t}|g|_\infty\left(1 + \int_H |y|^2\lambda(dy)\right), \tag{5.97}$$

for any $g \in UC_b(\mathbb{H}_s)$. Moreover, we know that, for any given initial law λ, there exists a subsequence $\{N_k^{'}\}_k$ such that $Y_{N_k^{'}}$ converges in distribution in $C(0, T; \mathbb{H}_s)$ to the law of Y. Recall that the family μ_N is tight in H during the proof of Theorem 3.1. Hence, for a subsequence $\{N_k\}_k$ of $\{N_k^{'}\}_k$, μ_{N_k} converges to a stationary measure $\mu^{(\lambda,\{N_k\})}$. $\mu^{(\lambda,\{N_k\})}$ means that μ depends on λ and $\{N_k\}$. Taking the limit in both sides of (5.47), we have

$$\left|\mathbb{E}_\lambda\Big(g(Y(t))\Big) - \int_H g(y)\mu^{(\lambda,\{N_k\})}(dy)\right| \le Ce^{-\gamma t}|g|_\infty\left(1 + \int_H |y|^2\lambda(dy)\right). \tag{5.98}$$

Thus, (5.97) is established with $\mu = \mu^{(\lambda,\{N_k\})}$. ■

In the last part, we want to prove that $\mu^{(\lambda,\{N_k\})}$ is independent on λ and $\{N_k\}$. First of all, Ψ in (2.31) satisfies the conditions required by [6], so for any initial value $y \in V$, there exists a unique global strong solution $Y(t, y)$.

Proof of Corollary 3.3 For any initial value $y \in V$, denote by $Y(t, y)$ is the unique global strong solution and its invariant measure is $\mu^{(\delta_y)}$, which does not depend on $\{N_k\}$. We claim that $\mu^{(\delta_y)} \equiv \mu^V$ for any $y \in V$, that is, the invariant measure μ^V for the strong solution is independent of the initial data $y \in V$. Indeed, suppose $y_i \in V (i = 1, 2)$ are two different elements, $\mu^{(\delta_{y_1})}$ and $\mu^{(\delta_{y_2})}$ are the corresponding invariant measures. Consider

$$\begin{aligned}\left|\int_H g(y)\mu^{(\delta_{y_1})}(dy) - \int_H g(y)\mu^{(\delta_{y_2})}(dy)\right| \;\; &\le \;\; \left|\int_H g(y)\mu^{(\delta_{y_1})}(dy) - \mathbb{E}g(Y_t^{y_1})\right| \\ &+ \;\; \left|\mathbb{E}g(Y_t^{y_1}) - \mathbb{E}g(Y_t^{y_2})\right| + \left|\mathbb{E}g(Y_t^{y_2}) - \int_H g(y)\mu^{(\delta_{y_2})}(dy)\right| \\ &= \;\; I_1 + I_2 + I_3,\end{aligned} \tag{5.99}$$

we have

$$\begin{aligned}I_2 = \left|\mathbb{E}g(Y_t^{y_1}) - \mathbb{E}g(Y_t^{y_2})\right| \;\; &\le \;\; \left|\mathbb{E}g(Y_t^{y_1}) - \mathbb{E}g(Y_t^{N,y_1})\right| \\ &+ \;\; \left|\mathbb{E}g(Y_t^{N,y_1}) - \mathbb{E}g(Y_t^{N,y_2})\right| + \left|\mathbb{E}g(Y_t^{N,y_2}) - \mathbb{E}g(Y_t^{y_2})\right| \\ &= \;\; I_4 + I_5 + I_6.\end{aligned}$$

Thus,
$$\left|\int_H g(y)\mu^{(\delta_{y_1})}(dy) - \int_H g(y)\mu^{(\delta_{y_2})}(dy)\right| \le I_1 + I_3 + I_4 + I_5 + I_6.$$
By (5.95) and (5.98), for any $\epsilon > 0$, there exists $t_0 > 0$ such that for any $t > t_0$
$$I_1 + I_3 + I_5 \le \epsilon \text{ uniformly for N.}$$
Fix $t > t_0$, and let $N \to \infty$, we obtain $I_4 + I_6 \to 0$. Thus,
$$\mu^{(\delta_y)} = \mu^V \quad \text{for any y} \in \text{V}. \tag{5.100}$$

Now, we are ready to show that all weak solutions which are limits of Galerkin approximation share the same invariant measure. Let $\lambda = \delta_{y_0}$, $y_0 \in H$, choosing $y_1 \in V$, we consider
$$\begin{aligned}
\left|\int_H g(y)\mu^{(\delta_{y_0},\{N_k\})}(dy) - \int_H g(y)\mu^V(dy)\right| \le\ & \left|\int_H g(y)\mu^{(\delta_{y_0},\{N_k\})}(dy) - \int_{P_{N_k}H} g(y)\mu_{N_k}^{(\delta_{y_0})}(dy)\right| \\
+\ & \left|\int_{P_{N_k}H} g(y)\mu_{N_k}^{(\delta_{y_0})}(dy) - \mathbb{E}g(Y_t^{N_k,y_0})\right| + \left|\mathbb{E}g(Y_t^{N_k,y_0}) - \mathbb{E}g(Y_t^{N_k,y_1})\right| \\
+\ & \left|\mathbb{E}g(Y_t^{N_k,y_1}) - \mathbb{E}g(Y_t^{y_1})\right| + \left|\mathbb{E}g(Y_t^{y_1}) - \int_H g(y)\mu^V(dy)\right| \\
=\ & K_1 + K_2 + K_3 + K_4 + K_5.
\end{aligned}$$

By (5.95), (5.96), (5.98) and (5.100), when $t \to \infty$, we have $K_2 + K_3 + K_4 + K_5 \to 0$ uniformly for N_k, and then, letting $N_k \to \infty$, we obtain $K_1 \to 0$. Thus
$$\int_H g(y)\mu^{(\delta_{y_0},\{N_k\})}(dy) = \int_H g(y)\mu^V(dy),$$
which implies $\mu^{(\delta_{y_0},\{N_k\})}$ is independent of δ_{y_0} and $\{N_k\}$. Moreover, we have $\mu^{(\delta_{y_0},\{N_k\})} = \mu^V$. ■

Acknowledgements This work was supported by National Natural Science Foundation of China (NSFC) (No. 11431014, No. 11371041, No. 11401557, No. 11271356), the Fundamental Research Funds for the Central Universities (No. 0010000048), Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences(No. 2008DP173182), and the applied mathematical research for the important strategic demand of China in information science and related fields(No. 2011CB808000).

References

[1] R.A. Adams: *Sobolev Space.* New York: Academic Press, 1975

[2] C. Cao, E.S. Titi: *Global well-posedness of the three-dimensional viscous primitive equations of large-scale ocean and atmosphere dynamics.* Ann. of Math. 166, 245-267 (2007)

[3] C. Cao, J. Li, E.S. Titi: *Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity*. Arch. Rational Mech. Anal. 214 35-76 (2014).

[4] G. Da Prato, J. Zabczyk: *Ergodicity for Inifinite Dimensional Systems.* In: London Mathematical Society Lecture Notes,n.229, Cambridge University Press, Cambridge (1996)

[5] A. Debussche, N. Glatt-Holtz, R. Temam: *Local martingale and pathwise solutions for an abstract fluids model.* Phys. D 240 , no. 14-15, 1123-1144 (2011)

[6] A. Debussche, N. Glatt-Holtz, R. Temam, M. Ziane: *Global existence and regularity for the 3D stochastic primitive equations of the ocean and atmosphere with multiplicative white noise.* Nonlinearity, 25, 2093-2118 (2012)

[7] F. Flandoli, D. Gatarek: *Martingale and stationary solutions for stochastic Navier-Stokes equations.* Probab. Theory Related Fields 102, 367-391 (1995)

[8] C. Frankignoul, K. Hasselmann: *Stochastic climate models, Part II: Application to sea-surface temperature anomalies and thermocline variability.* Tellus 29, 289-305 (1977)

[9] N. Glatt-Holtz, I. Kukavica, V. Vicol, M. Ziane: *Existence and regularity of invariant measures for the three dimensional stochastic primitive equations.* arXiv:1311.4204v1

[10] F. Guillén – González, N. Masmoudi, M.A. Rodriguez-Bellido: *Anisotropic estimates and strong solutions for the primitive equations.* Diff. Int. Equ. 14, 1381-1408(2001)

[11] B. Guo, D. Huang: *3D stochastic primitive equations of the large-scale ocean: global well-posedness and attractors.* Comm. Math. Phys. 286, no. 2, 697-723 (2009)

[12] M. Hairer: *Exponential Mixing Properties of stochastic PDEs through Asymptotic coupling.* Prob. Theory Rel. Fields 124(3), 345-380 (2002)

[13] C. Hu, R. Temam, and M. Ziane: *The primitive equations on the large scale ocean under the small depth hypothesis.* Discrete Contin. Dynam. Systems 9, 97-131 (2003)

[14] S. Kuksin: *On exponential convergence to a stationary measure for nonlinear PDEs*. Partial differential equations, 161-176 (2002)

[15] S. Kuksin, A. Shirikyan: *Ergodicity for the randomly forced 2D Navier-Stokes equations.* Math. Phys. Anal. Geom. 4, 147-195 (2001)

[16] S. Kuksin, A. Shirikyan: *A coupling approach to randomly forced PDE's I.* Comm. Math. Phys. 221, no. 2, 351-366 (2001)

[17] S. Kuksin, A. Piatnitski, A. Shirikyan: *A coupling approach to randomly forced PDE's II.* Comm. Math. Phys. 230(1), 81-85 (2002)

[18] J.L. Lions, R. Temam, S. Wang: *New formulations of the primitive equations of atmosphere and applications.* Nonlinearity 5, 237-288 (1992)

[19] J.L. Lions, R. Temam, S. Wang: *On the equations of the large scale ocean.* Nonlinearity 5, 1007-1053 (1992)

[20] J.L. Lions, R. Temam, S. Wang: *Models of the coupled atmosphere and ocean.* Computational Mechanics Advance 1, 1-54 (1993)

[21] J.L. Lions, R. Temam, S. Wang: *Mathematical theory for the coupled atmosphere-ocean models.* J. Math. Pures Appl. 74, 105-163 (1995)

[22] J. Mattingly: *Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics.* Comm. Math. Phys. 230, 421-462 (2002)

[23] U. Mikolajewicz, E. Maier-Reimer: *Internal secular variability in an OGCM.* Climate Dyn. 4, 145-156 (1990)

[24] T. Tachim Medjo: *The exponential behavior of the stochastic three-dimensional primitive equations with multiplicative noise.* Nonlinear Anal. Real World Appl. 12, no. 2, 799-810 (2011)

[25] D. Nualart: *Malliavin Calculus and related topic.* Probability and its Applications, 93B Newyork: Springer, 1995

[26] C. Odasso: *Exponential mixing for the 3D stochastic Navier-Stokes equations.* Comm. Math. Phys. 270, no. 1, 109-139 (2007)

[27] C. Odasso: *Ergodicity for the stochastic Complex Ginzburg-Landau equations.* Ann. Inst. H. PoincarĺȩProbab. Statist. 42, no. 4, 417-454 (2006)

[28] O.M. Phillips: *On the generation of waves by turbulent winds.* J. Fluid Mech. 2, 417-445 (1957)

[29] R. Temam: *Navier-Stokes equations and nonlinear functional analysis.* Second edition. CBMS-NSF Regional Conference Series in Applied Mathematics, 66. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995